

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N. Y.



Institute of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

AFOSR-1380

MF-17

THE INITIAL VALUE PROBLEM,
SOUND PROPAGATION, AND MODELING
IN KINETIC THEORY

Lawrence Sirovich

September 15, 1961

NEW YORK UNIVERSITY

11
C.2



UNCLASSIFIED

Magneto-Fluid Dynamics Division
Institute of Mathematical Sciences
New York University

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
4 Washington Place, New York 3, N. Y.

AFOSR-1380

MF-17

THE INITIAL VALUE PROBLEM,
SOUND PROPAGATION, AND MODELING
IN KINETIC THEORY

Lawrence Sirovich

September 15, 1961

Contract No. AF-49(638)1006
Air Force Office of Scientific Research

UNCLASSIFIED

Abstract

The one dimensional initial value problem of a monatomic single component gas is considered. Using the linearized Boltzmann equation the dispersion relation is studied. In addition to the usual gas dynamic sound waves one finds an infinity of decaying propagating waves. The phenomenon naturally exhibits itself as a sequence of epochs, the last stage of which is hydrodynamic.

With reference to the same problem macroscopic equations such as Euler, Navier-Stokes, Burnett, Grad's moments equations, etc., are considered. In addition the recently considered "kinetic models" of Gross et al. are applied to the problem. These various formulations are critically analyzed and compared with each other and with the Boltzmann analysis. Lastly, several alternate molecular and macroscopic equations are offered which remedy some of the shortcomings which appear in the above mentioned approximate theories.

Table of Contents

	page
Abstract	2
Section	
1. Introduction	4
2. The Linearized Boltzmann Equation	10
3. Eigentheory	27
4. Some Remarks on Solutions of the Boltzmann Equations	45
5. Kinetic and Macroscopic Models	51
Appendix: Two Lemmas on Determinants	78
Tables	81
Figures	86

The Initial Value Problem, Sound Propagation, and Modeling in Kinetic Theory

1. Introduction

The mathematical difficulties encountered in dealing with the Boltzmann equation have motivated the creation and study of many approximate formulations. In this class of approximate formulations are the Euler equations, Navier-Stokes equations, Burnett equations¹, the moments equations of H. Grad², and the interpolated equations of H. Grad.³ A feature shared by each of these systems is that they may be considered as having been extracted from the Boltzmann equation by a process which regards a flow as being smooth with respect to the mean-free-path as a length scale, and smooth with respect to the collision frequency

-
1. S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press (1952).

A fluid dynamicist might object to classifying the Navier-Stokes equations as an approximation to the Boltzmann equation. His objection is warranted by the fact that the Navier-Stokes equations apply to a wider range of fluids than does the Boltzmann equation. We, however, restrict our attention to gases, in which case the Euler equations, Navier-Stokes equations, and Burnett equations can be thought of as being extracted from the Boltzmann equation by the Chapman-Enskog procedure as given, for instance, in the cited reference.

2. H. Grad, Comm. Pure and Appl. Math. 2, 331 (1949).
3. H. Grad, "Theory of Rarefied Gases", in Rarefied Gas Dynamics, ed. F. M. Devienne (Pergamon Press, London, 1960), p. 100.

Grad gives a technique whereby the thirteen moments equations in a quick way give the Navier-Stokes equation. This method, which Grad calls interpolation, is generalized by him to any system of moments equations.

as a time scale. This presumably excludes from consideration such "kinetic" boundary layers as the region adjacent to a wall (not to be confused with the fluid mechanical boundary layer which, in general, is much thicker), the interior of a strong shock, and layers of large shear. There is, in addition to the spatial boundary layers just mentioned, the associated temporal boundary layer. This boundary layer arises from the fact that the above mentioned techniques for deriving fluid equations are presumably not applicable at all for times less than the mean collision time.

An entirely different class of approximate formulations has its origin in the model equation associated with Bhatnager, Gross, and Krook⁴, and Welander.⁵ This was subsequently generalized by Gross and Jackson⁶ to give a wide range of model equations. Whereas the approximate formulations are equations in moments of the molecular distribution function, a model equation is an equation in the molecular distribution function itself. These model equations share many common features with the Boltzmann equation, and are sometimes more amenable to analysis. As will be shown, these model equations overcome some serious shortcomings of the macroscopic equations (and introduce others).

4. P. F. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. 94, 511 (1954).

5. P. Welander, Arkiv Fysik 7, No. 44 (1954).

6. E. P. Gross and E. A. Jackson, Phys. Fluids 2, 432 (1959).

One of the major objectives of this study is to relate the above mentioned approximations (and others) to the Boltzmann equations and to each other. A precise analysis would involve estimates on the solutions of all systems under consideration and a subsequent comparison of them. For obvious reasons such a plan is not feasible. Instead we will attempt a more modest and in some respects a more revealing approach. The non-linear theory will be discarded and we consider only the initial value problem of the one dimensional linearized theory in the absence of boundaries. Our analysis will be performed exclusively for this elementary but by no means simple problem. We will consider the latter problem as exhaustively as possible, and this will take up a considerable part of this report. The approximate theories will then be applied to the same initial value problem. Only when all of these calculations have been performed will we be able to connect the various theories amongst themselves and to the Boltzmann equation. As a word of caution it should be noted that the various interrelations which are made can only justifiably be made for the particular case under study -- it is tempting none-the-less to consider these results as being more general. On the other hand, any inadequacy of a set of equations which is pointed out is an inadequacy in general, although it may be less important in another problem.

The analysis of the Boltzmann equation has been based on the

investigations of sound propagation made by C. S. Wang Chang and G. E. Uhlenbeck⁷, and the later, more general developments in linearized kinetic theory made by H. N. Mott-Smith.⁸ The author is in no small way indebted to these workers as is evidenced by the frequent reference to their work.

Before going on to the analysis it will be advantageous to outline some of the results which are later found. The evolution in time of the solution from the initial time is in a natural manner divided into epochs. With each epoch is associated a propagating mode or a diffusing mode or both, there being an infinity of each type. As time evolves, clusters of modes become washed out of the picture in the sense that they have a shorter exponential damping time than those remaining. In the final stage there is just one cluster of modes compared to which all other modes are asymptotically small. This is the usual hydrodynamic regime.

Macroscopic equations such as the Navier-Stokes equations and Burnett equations only furnish a description of the hydrodynamic stage of the initial value problem. Higher approximations such as Burnett and any other that might come from a Chapman-Enskog procedure attempt to give a finer description of this last stage.

The moments equations of H. Grad², on the other hand, attempt

7. C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in a Monatomic Gas", Univ. of Mich., Engr. Rep. (1952).

In this highly original and stimulating report, the authors make the first general attack on sound propagation, by means of the Boltzmann equation. The analysis in Section 2 is largely based on the work of this report, and that of the following cited work of H. M. Mott-Smith.

8. H. M. Mott-Smith, "A New Approach to the Kinetic Theory of Gases", Lincoln Laboratory, MIT (1954).

to take into consideration some of the modes that do occur prior to the hydrodynamic stage. However these equations have two shortcomings. First of all any moments system has a bound on the absorption of the system. This is contrary to the results gotten from Boltzmann equation, which has modes of arbitrary high absorption, and suggests moreover that any mode approaches unbounded absorption with high frequencies. As is shown the interpolation scheme of H. Grad³ overcomes this by changing the type of the equation. For instance interpolation sends the thirteen moments equations into the Navier-Stokes equations. The second drawback to the moment methods involves the onset of propagation for the non-hydrodynamic mode. They can incorrectly predict a threshold frequency at which extraordinary propagation occurs. The latter effect is connected to the moments one chooses in a description. Alternative systems of equations can be introduced which overcome these difficulties.

The equations falling under the title of kinetic models also take into consideration the presence of other modes. It does this by lumping together an infinity of modes into a single cluster and giving a more accurate description of a few modes. The single relaxation model of Bhatnager, Gross and Krook for instance gives only a qualitatively correct description of the hydrodynamic stage. The kinetic models which occur in the literature give an incorrect picture of the onset of the extraordinary sound propagations. They do this for essentially the same reason as this occurs in the moments equations.

Alternate models are easily developed which overcome this, and without a loss of simplicity. Moreover kinetic models also share with the moments systems the property of placing a bound on the absorption of the system. On the other hand within the confines of this absorption band the kinetic models give at least a qualitatively more correct description of the spectrum of the linearized Boltzmann equation. Furthermore kinetic models can be applied to highly rarefied gases. For instance they can be used in a Knudsen iteration procedure. Macroscopic equations by their very nature cannot be so applied.

In regards to the initial value problem, kinetic models lie between the Boltzmann equation analysis and a macroscopic description. In fact by choosing a sufficiently detailed kinetic model any macroscopic system of equations can be extracted from it without any resort to the Boltzmann equation.

Acknowledgment: The author is grateful to C. A. Sirovich for many helpful mathematical discussions.

2. The Linearized Boltzmann Equation

The Boltzmann equation of a single component gas is given by:

$$(2.1) \quad \left(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial x} \right) f = \frac{1}{m} \int (f'_* f' - f f_*) B(\theta, U) d\epsilon \, d\theta \, d\xi_*$$

where

$$(2.2) \quad U = |\xi_* - \xi|.$$

The notation is that usually found ^{1, 9}. $f(x, \xi, t)$ is the mass density in (x, ξ, t) space, $\xi = (\xi_1, \xi_2, \xi_3)$ denotes the molecular velocity and ξ_* the velocity of a particle after a collision. A star subscript is used instead of 1, in order to avoid confusion with ξ_1 . The function $B(\theta, U)$ under the integral, is representative of the intermolecular force, with θ and ϵ giving the orientation of a collision.

We will consider a gas which is slightly removed from a state of equilibrium, given by

$$(2.3) \quad f = f^0 = \frac{\rho_0}{(2\pi RT_0)^{3/2}} e^{-\xi^2/2RT_0}$$

and which is itself a solution to equation (2.1). ρ_0 and T_0 are, respectively, the constant density and temperature. A small departure from equilibrium is meant in the sense that:

$$(2.4) \quad \begin{cases} f = f^0(1 + g) \\ g^2 \ll 1. \end{cases}$$

9. H. Grad, "Principles of the Kinetic Theory of Gases," Handbuch der Physik, vol. 12 (Springer, 1958)

For a problem such as sound propagation, this approximation may be used with confidence. Introducing (2.4) into the Boltzmann equation, we get on linearization

$$(2.5) \quad \left(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial x} \right) g = \frac{1}{m} \int f_*^0 (g' + g'_* - g - g_*) B(\theta, U) d\varepsilon d\theta d\xi_*$$

We can show some general properties of the integral operator of equation (2.5) by introducing,

$$(2.6) \quad J(\phi) = \int f^0 f_*^0 [\phi] B(\theta, U) d\varepsilon d\theta d\xi_*,$$

with

$$(2.7) \quad [\phi] = \phi' + \phi'_* - \phi - \phi_*.$$

In this notation it can be shown¹⁰

$$(2.8) \quad \int \psi J(\phi) d\xi = \frac{-1}{4} \int f^0 f_*^0 [\psi] [\phi] B d\varepsilon d\theta d\xi_* d\xi.$$

We introduce the eigenfunctions, ϕ , of the linear operator J , by writing

$$(2.9) \quad J(\phi) = \lambda f^0 \phi.$$

The Maxwellian f^0 is extracted from the eigenfunction mostly as a matter of convenience. As Wang Chang and Uhlenbeck⁷ have shown, the eigenvalues are real and non-positive. For from equations (2.8) and (2.9) we have,

10. This may be found in Ref. 1, p. 67 and in Ref. 12, p. 237.

$$(2.10) \quad \lambda = - \frac{1}{4} \frac{\int f^0 f_1^0 |\phi|^2 B(\theta, U) d\varepsilon d\theta d\xi_* d\xi}{\int f^0 |\phi|^2 d\xi}$$

and since $B(\theta, U)$ is positive¹ the eigenvalue λ is non-positive. If ϕ is either 1, ξ , or ξ^2 , λ is zero. This follows from the conservation of momentum, energy and the number of particles, in a two body collision.

Further from (2.8) we get the orthogonality of eigenfunctions with respect to the weight function f_0 . For eigenvalues μ, ν such that $\mu \neq \nu$, we have

$$(2.11) \quad (\mu - \nu) \int f^0 \Phi \psi d\xi = 0$$

where Φ and ψ are the corresponding eigenfunctions. If we choose to represent the linear operator $J(\phi)$ as

$$(2.12) \quad J(\phi) = \int K(\xi, \xi_*) \phi(\xi_*) d\xi_*$$

we immediately see from (2.8) that K is a symmetric kernel. The latter property is usually shown by construction¹¹.

Without loss of generality, we may take the eigenfunctions at J to be normalized to unity, and we further presume that the eigenfunctions, which we denote by $\{\phi_i\}$, form a complete orthonormal set. We expand the perturbed distri-

11. Ref. 1, p. 129.

bution function in this set,

$$(2.13) \quad g = \sum_{n=0}^{\infty} a_n \phi_n$$

where the coefficients are given by

$$(2.14) \quad a_n = \int f^0 g \bar{\phi}_n d\xi \quad ,$$

where the bar refers to the complex conjugate. As we will see later there seem to be several "sensible" ways in which to order the eigenfunctions. We will for the moment think of them as being ordered in relation to the ascending order of the magnitudes of their corresponding eigenvalues. If the expansion (2.13) is substituted into the linearized equation (2.5) we have

$$(2.15) \quad \left(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial x} \right) \sum_{n=0}^{\infty} a_n \phi_n = \sum_{n=5}^{\infty} a_n \lambda_n \phi_n$$

where we have denoted the eigenvalues by λ_n . One should notice that the summation of the right hand side starts with $n = 5$, since, as already mentioned above, there are five zero eigenvalues. By making use of the orthogonality property we may represent equation (2.15) by

$$(2.16) \quad \frac{\partial}{\partial t} a_i + b_{ij} \cdot \frac{\partial}{\partial x} a_j - \lambda_{ij} a_j = 0$$

where

$$(2.17) \quad b_{ij} = \overline{b_{ji}} = \int r^0 \phi_i \overline{\phi_j} \xi \, d\xi$$

and

$$(2.18) \quad \lambda_{ij} = (\lambda_i) \delta_{ij}$$

The parenthesis in the last relation signifies that the summation convention, which is used everywhere else, is not to be used. Since the matrix b_{ij} ¹², as is shown by (2.17) is hermitian, the system of partial differential equations (2.16) is of the symmetric hyperbolic type discussed in the literature¹³. The system, however, has an infinite number of variables and equations. In later sections use of the hyperbolicity will be made, however it will not extensively be applied in this study. A further remark in this connection is worth making. The "energy" which is usually defined when talking of symmetric hyperbolic equations is the sum,

$$a_i \overline{a_i}$$

In this case however this turns out to be the linearized H-function¹⁴.

We now leave these very general remarks and for the moment

12. b_{ij} should be thought of as a three-component vector, each component of which has the two scripts ij .

13. K. O. Friedrichs, Comm. Pure Appl. Math. 7 (1954); 11 (1958).

14. This has already been pointed out by H. Grad, ref. 3, p. 109.

at least take $B = B(\theta)$. It is known^{1,2,4} that Maxwell molecules (i.e., molecules which behave with an intermolecular force law which is proportional to the inverse fifth power of the distance between particles) behave according to such relation. As we will see, this assumption greatly simplifies the analysis.

For Maxwell molecules then, equation (2.5) becomes

$$(2.19) \quad \left(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial x} \right) g = L(g) \\ = \frac{1}{m} \int f_*^0 (g' + g'_* - g - g_*) B(\theta) d\varepsilon d\theta d\xi_* .$$

The eigenfunctions of $L(g)$ are the complete set of functions given by the product of Sonine polynomials with spherical harmonics. Although the facility offered by these functions was noticed for instance by Burnett¹⁵, the eigenfunction property was first pointed out in general by Wang Chang and Uhlenbeck in their pioneering report.¹⁶

Modulo normalization, the eigenfunctions are represented by

$$(2.20) \quad \bar{\Phi}_{r\ell m} = S_{\ell+1/2}(\hat{\xi}^2/2) \hat{\xi}^\ell P_\ell^m(\cos \psi) g^{im\phi}$$

15. D. Burnett, Proc. Lond. Math. Soc. 40, 382 (1935).

16. Wang Chang and Uhlenbeck show this in Ref. 6. It was shown independently, although somewhat later, by L. Waldmann, "Transporterscheinungen von mittleren Druck", Handbuch der Physik, vol. 12 (Springer, 1958). An elegant and simple proof of this property is given by H. Grad in Ref. 9.

where the molecular velocity is normalized with respect to the equilibrium thermal speed,

$$(2.21) \quad \hat{\xi} = \xi / \sqrt{RT_0} \quad .$$

Referring once again to a general molecular force, Wang Chang and Uhlenbeck⁷ showed that the eigenfunctions of equation (2.6) always have the angular dependance in velocity space given by (2.20). One should note that our normalization of the velocity is slightly different than that chosen by Mott-Smith⁸, and Wang Chang and Uhlenbeck⁷ (we use $\sqrt{RT_0}$ and they $\sqrt{2RT_0}$). This results in slight variations in the formulae which are given on the next few pages. The Sonine polynomials can be represented by

$$(2.22) \quad S_m^n(x) = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left\{ (1-s)^{m+n} e^{sx} \right\} \Big|_{s=0} \quad .$$

For a discussion of a Sonine polynomials, and Legendre polynomials P_ℓ^m , one is referred to one of the standard works¹⁷.

The eigenfunctions have the orthogonality property given by

$$(2.23) \quad \frac{1}{(2\pi)^{3/2}} \int e^{-\hat{\xi}^2/2} \phi_{rlm} \phi_{r'l'm'} d\hat{\xi} = \delta_{rr'} \delta_{\ell\ell'} \delta_{mm'} N_{rlm}$$

where,

17. For instance, W. Magnus and F. Oberhettiger, Formulas and Theorems for the Special Functions of Mathematical Physics, (Chelsea, 1948).

$$(2.24) \quad N_{r\ell m} = \frac{2^{\ell+1} \overline{\Gamma}(r+\ell+3/2)}{\sqrt{\pi} r! (2\ell+1)} \cdot \frac{(\ell+|m|)!}{(\ell-|m|)!}$$

where the bar again refers to the complex conjugate. Of great aid in the succeeding work is the following recursion relation

$$(2.25) \quad (2\ell+1)\hat{\xi}_1 \phi_{r\ell m} = (\ell-m+1)[\phi_{r,\ell+1,m} - \phi_{r-\ell+1,m}] \\ + 2(\ell+m)[(r+\ell+\frac{1}{2})\phi_{r,\ell-1,m} - (r+1)\phi_{r+1,\ell-1,m}] \quad .$$

On expanding the perturbed distribution function in terms of the eigenfunctions, $\phi_{r\ell m}$, we get,

$$(2.26) \quad g = \sum a_{r\ell m} \phi_{r\ell m}$$

with the relation for the coefficients given by

$$(2.27) \quad a_{r\ell m} = \frac{1}{\beta N_{r\ell m}} \int f^0 g \bar{\phi}_{r\ell m} d\xi \quad .$$

Note that all coefficients are immediately dimensionless.

Next denoting the eigenvalues of L by $\lambda_{r\ell m}$, and substituting the expansion into the linearized equation (2.6), we get,

$$(2.28) \quad \left\{ \begin{array}{l} L\phi_{rlm} = \lambda_{rlm} \phi_{rlm} \\ \left(\frac{\partial}{\partial t'} + \hat{\xi} \cdot \frac{\partial}{\partial x'} \right) \sum a_{rlm} \phi_{rlm} = \sum a_{rlm} \lambda'_{rlm} \phi_{rlm} \end{array} \right. .$$

Wang Chang and Uhlenbeck⁷, have shown that λ_{rlm} , regardless of the molecular model, is independent of the index m .

Therefore each eigenvalue is at least $(2l+1)$ -fold degenerate. A further degeneracy for Maxwell molecules is pointed out in the following section. The latter equation has been made dimensionless by means of a frequency ν , in the following way,

$$(2.29) \quad \left\{ \begin{array}{l} \nu t = t' \\ \frac{x\nu}{\sqrt{RT_0}} = x' \\ \frac{\lambda_{rlm}}{\nu} = \lambda'_{rlm} \end{array} \right.$$

We will leave the frequency ν unspecified for the present.

Our considerations will now be restricted to one-dimensional unsteady flows. This will change the convective terms to $\xi_1 \frac{\partial}{\partial x}$. The variable x now represents a single space coordinate. With this restriction we see that the elements of the coefficient matrix b_{ij} , defined by equation (2.17) is easily evaluated by means of the recursion relation (2.25). Examination of this recursion relation shows that the

dependence indicated by the index m drops identically from equation (2.28). From this we can conclude that if the initial data of a problem is homogeneous with respect to the index m , then the solution will not develop an index m in time. In particular if the initial disturbance is isotropic in the plane normal to the x direction, we can set $m = 0$ in the above. This latter case can be regarded as appropriate for the study of sound propagation and will be the case which is taken up here. However it should not be presumed general enough for the study of all one-dimensional problems. For instance, the Rayleigh problem is strictly one dimensional, but nevertheless requires shear moments for an analysis. Henceforth we restrict attention to the $m = 0$ case and denote the eigenfunctions, and eigenvalues, respectively, by $\phi_{r\ell}$ and $\lambda_{r\ell}$.

Rather than use $\phi_{r\ell}$ we will with Wang Chang and Uhlenbeck⁷ use the normalized eigenfunctions,

$$(2.30) \quad \psi_{r\ell} = \frac{\phi_{r\ell}}{\sqrt{\frac{2^{\ell+1}}{\pi^{1/2}} \frac{(r+\ell+3/2)}{r!(2\ell+1)}}}$$

with the property ¹⁸,

18. The eigenfunctions are real because of the suppression of the index m .

$$(2.31) \quad \int d\hat{\xi} \, \omega \, \psi_{r\ell} \psi_{r'\ell'} = \delta_{rr'} \delta_{\ell\ell'}$$

with

$$(2.32) \quad \omega = \frac{1}{(2\pi)^{3/2}} e^{-\hat{\xi}^2/2}.$$

The recursion relation now has the form,

$$(2.33) \quad (2\ell+1)\xi_1 \psi_{r\ell} = (\ell+1) \left\{ \psi_{r\ell+1} \sqrt{\frac{(r\ell+3/2)2}{(2\ell+1)(2\ell+3)}} - \right. \\ \left. \psi_{r-1,\ell+1} \sqrt{\frac{2r}{(2\ell+3)(2\ell+1)}} + \ell \left\{ \psi_{r\ell-1} \sqrt{\frac{(r+\ell+1/2)2}{(2\ell+1)(2\ell-1)}} - \right. \right. \\ \left. \left. \psi_{r+1,\ell-1} \sqrt{\frac{2(r+1)}{(2\ell-1)(2\ell+1)}} \right\} \right\}.$$

In the new notation the perturbed distribution function g is

$$(2.34) \quad g = \sum_{r\ell} b_{r\ell} \psi_{r\ell}$$

with

$$(2.35) \quad b_{r\ell} = \int \omega \, g \, \psi_{r\ell} \, d\hat{\xi}.$$

And the linearized equation (2.15) now becomes,

$$(2.36) \quad \left(\frac{\partial}{\partial t'} + \hat{\xi}_1 \frac{\partial}{\partial x'} \right) \sum_{r\ell} b_{r\ell} \psi_{r\ell} = \sum_{r\ell} b_{r\ell} \lambda'_{r\ell} \psi_{r\ell}.$$

Several examples of the eigenfunctions and their associated coefficients are:

$$(2.37) \quad \left\{ \begin{array}{ll} \psi_{00} = 1 & b_{00} = \hat{\rho} \\ \psi_{01} = \hat{\xi}_1 & b_{01} = \hat{u}_1 \\ \psi_{10} = \sqrt{\frac{3}{2}} \left(1 - \frac{\xi^2}{3}\right) & b_{10} = -\sqrt{\frac{3}{2}} \hat{T} \\ \psi_{02} = \sqrt{\frac{3}{2}} \left(\hat{\xi}_1^2 - \frac{\xi^2}{3}\right) & b_{02} = \sqrt{\frac{3}{2}} \hat{p}_{11} \\ \psi_{11} = \sqrt{\frac{5}{2}} \hat{\xi}_1 \left(1 - \frac{\xi^2}{3}\right) & b_{11} = -\sqrt{\frac{2}{5}} \hat{S} \end{array} \right.$$

The quantities on the right have been made dimensionless with respect to $T_0, \sqrt{RT_0}, \rho_0$, and are respectively, the perturbed density, velocity, temperature, stress and heat conduction. In the present notation the "gas law" identity is

$$(2.38) \quad \hat{p} = \hat{\rho} + \hat{T}.$$

Here \hat{p} is of course the dimensionless pressure, and it will be eliminated in what follows by means of relation (2.38).

We now wish to generate differential equations for the determination of the coefficients, $b_{r\ell}$. First substituting the recursion relation (2.33) into equation (2.36) we get,

$$\begin{aligned}
(2.39) \quad 0 = & \sum_{r,\ell} \psi_{r\ell} \left(\frac{\partial}{\partial t^1} - \lambda_{r\ell} \right) b_{r\ell} \\
& + \frac{\partial}{\partial x^1} \sum b_{r\ell} \left[(1+\ell) \left\{ \psi_{r,\ell+1} \sqrt{\frac{(r+\ell+3/2)^2}{(2\ell+1)(2\ell+3)}} \right. \right. \\
& - \psi_{r-1,\ell+1} \sqrt{\frac{2r}{(2\ell+3)(2\ell+1)}} \left. \right\} + \ell \left\{ \psi_{r,\ell-1} \sqrt{\frac{(r+\ell+1/2)^2}{(2\ell+1)(2\ell-1)}} \right. \\
& \left. \left. - \psi_{r+\ell,\ell-1} \sqrt{\frac{2(r+1)}{(2\ell-1)(2\ell+1)}} \right\} \right].
\end{aligned}$$

And finally operating on the above with

$$(2.40) \quad \int d\hat{\xi} \, \omega \, \psi_{\mu\nu}$$

we get the infinite system of differential equations governing the coefficients, $b_{\mu\nu}$.

$$\begin{aligned}
(2.41) \quad 0 = & \left(\frac{\partial}{\partial t^1} - \lambda'_{\mu\nu} \right) b_{\mu\nu} + \frac{\partial}{\partial x^1} \left[v \left(b_{\mu,\nu-1} \sqrt{\frac{(\mu+\nu+1/2)^2}{(2\nu-1)(2\nu+1)}} \right. \right. \\
& - b_{\mu+1,\nu-1} \sqrt{\frac{2\mu+2}{(2\nu-1)(2\nu+1)}} \left. \right) \\
& + (\nu+1) \left(b_{\mu,\nu+1} \sqrt{\frac{(\mu+\nu+3/2)^2}{(2\nu+2)(2\nu+1)}} \right. \\
& \left. \left. - b_{\mu-1,\nu+1} \sqrt{\frac{2\mu}{(2\nu+3)(2\nu+1)}} \right) \right].
\end{aligned}$$

As will be pointed out in the next section, the smallest

non-zero eigenvalue, in magnitude (all non-zero eigenvalues are < 0 , see (2.10)) is λ_{11} and we normalize with respect to it, i.e., we take

$$(2.42) \quad |\lambda_{11}| = \nu$$

in the normalization given in (2.29). Using the relations given in (2.38) and the fact that the first three eigenvalues vanish, the first three equations of (2.41) are

$$(2.43) \quad \frac{\partial}{\partial t} \hat{\rho} + \frac{\partial}{\partial x} \hat{u} = 0$$

$$(2.44) \quad \frac{\partial \hat{u}}{\partial t} + \frac{\partial}{\partial x} \hat{p}_{11} + \frac{\partial}{\partial x} (\hat{\rho} + \hat{T}) = 0$$

$$(2.45) \quad \frac{\partial \hat{T}}{\partial t} + \frac{2}{3} \frac{\partial}{\partial x} \hat{s}_1 + \frac{2}{3} \frac{\partial}{\partial x} \hat{u}_1 = 0.$$

These will be recognised as the linearized and dimensionless conservation of mass, momentum, and energy equations.

As is well-known this system is not determinate, it contains more unknowns than equations. Which is in fact the case for any finite collection of the equations of (2.41). If the system given in (2.43-2.45) is terminated by excluding the quantities which do not have a time derivative we have, on returning to the "b" notation,

$$(2.46) \quad \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\sqrt{2/3} \\ 0 & \sqrt{2/3} & 0 \end{pmatrix} \frac{\partial}{\partial x} \right] \begin{bmatrix} b_{00} \\ b_{01} \\ b_{10} \end{bmatrix} = 0$$

These are just the inviscid equations of gas dynamics, but arranged such that the coefficient matrix of the space derivative is symmetric. This as was shown earlier is a general property of any truncated system derived from the full set of (2.41).

It is clear that the ordering of the coefficients $b_{r\ell}$ is arbitrary. A complete solution of (2.41) will naturally be independent of the ordering, but it goes without saying that a complete solution is not feasible. On the other hand an approximate solution, for instance, one gotten by the truncation (2.41) will often be strongly dependent on the ordering of the $b_{r\ell}$. In fact various orderings are appropriate for different phenomena. This will become more evident as we proceed. We will, however, now indicate two orderings which seem natural and as we will see, are appropriate to different situations.

The first ordering is that which is almost universally chosen, according to the degree of the polynomial $\psi_{r\ell}$. We see from the expression given in (2.20) that $\psi_{r\ell}$ is a polynomial of degree $2r+\ell$. The ordering of $b_{r\ell}$ can thus be chosen to correspond to the degree of $\psi_{r\ell}$ and subordered in ascending values of the index ℓ . For example

$$(2.47) \quad b_{00}, b_{01}, b_{10}, b_{02}, b_{11}, b_{03}, b_{20}, b_{12}, b_{04}, \dots$$

The other ordering which seems natural, is in correspondence

with the ascending magnitudes of the eigenvalues, $|\lambda_{r\ell}|$, as was introduced at the outset of this section.

Before leaving this section we one again take up the case of non-Maxwell molecules. Aside from Maxwell molecules, no molecular force law permits the explicit representation of its corresponding eigenfunctions. This is really less of a liability than it seems. In the next section we consider the spectrum of the entire linear Boltzmann operator. As will be seen we shall be succesful in obtaining certain power series representations of the spectrum. This power series development is crucially dependent upon the Maxwell molecule eigenfunctions exhibited. More specifically it depends on the recursion relation given by (2.25). This fortifies the natural tendency to expand the distribution function in terms of Sonine polynomials in spite of the fact that they are not in general eigenfunctions of the integral operator. In regards to the differential operator of the Boltzmann equation they are so to speak the natural functions to expand in.

The one dimensional form of equation (2.5) will now be considered. The previous argument given to disregard the shear moments still applies. We therefore use the expansion given by (2.34). This gives,

$$\begin{aligned}
(2.48) \quad & \left(\frac{\partial}{\partial t} + \hat{\xi}_1 \frac{\partial}{\partial \mathbf{x}^1} \right) \sum_{r\ell} b_{r\ell} \psi_{r\ell} \\
& = \sum_{r\ell} b_{r\ell} \int \omega_* [\psi_{r\ell}] \hat{B}(\theta, U) d\epsilon \, d\theta \, d\hat{\xi}_*
\end{aligned}$$

where $\hat{B}(\theta, U)$ has been normalized accordingly. On operating on (2.48) with (2.40) in order to generate moment equations we obtain the same differential operator as is found in equation (2.41). Instead of a single term appearing from the collision operator we now get,

$$(2.49) \quad \sum_r \lambda_{r\mu\nu} b_{rv} = \sum_r b_{rv} \int \omega_* \omega [\psi_{rv}] \psi_{\mu\nu} \hat{B}(\theta, U) d\epsilon \, d\theta \, d\hat{\xi}_* d\hat{\xi} .$$

The contraction of the second subscript follows from the remark following equation (2.21). For Maxwell molecules the $\lambda_{r\mu\nu}$ vanishes for $r \neq \mu$. Mott-Smith⁸ has considered $\lambda_{r\mu\nu}$ for a wide range of molecular force laws, and has shown that $\lambda_{r\mu\nu}$ is "almost independent of the force law, at least for small r, μ, ν ". It is in fact the case that several of these elements are independent of the force⁸. This being the case we can, for non-Maxwell molecules take with some confidence only the diagonal part of $\lambda_{\mu rv}$. One may easily show that this furnishes what Chapman and Cowling call the first approximation in the determination of the transport coefficients.

3. Eigentheory

This section will be devoted to the computation of the eigenvalues of the linearized Boltzmann operator; and the subsequent calculation of the eigenvalues of the differential-integro operator (2.5). The latter operation is equivalent to the determination of the spectrum as given by the dispersion relation.

From the previous section we may represent the eigenvalues of the integral operator by

$$(3.1) \quad \lambda_{r\ell} = \int d\xi \, \omega \, \psi_{r\ell} L(\psi_{r\ell}) .$$

All but the θ -integration in (3.1) can be carried out^{7,8} to get

$$(3.2) \quad \lambda_{r\ell} = 2\pi \int_0^\pi d\theta \, B(\theta) \left[\cos^{2r+\ell} \frac{\theta}{2} P_\ell(\cos \frac{\theta}{2}) \right. \\ \left. + \sin^{2r+\ell} \frac{\theta}{2} P_\ell(\sin \frac{\theta}{2}) - (1 + \delta_{r0} \delta_{r\ell}) \right] .$$

For small values of θ , $B(\theta)$ behaves as $\theta^{-3/2}$,^{1,7} so we see that individual terms of the integrand lead to divergent integrals. However, the combination of terms erases this divergence, as can be seen by inspection. We see from (3.2) that

$$(3.3) \quad \lambda_{r0} = \lambda_{r-1, 1} .$$

This degeneracy will play an important roll in the succeeding work. Wang Chang and Uhlenbeck⁷ have evaluated some of the eigenvalues and later Mott-Smith⁸ evaluated many more. A compilation

of this may be found in reference 17. Table I contains the collection of values of $\lambda'_{r\ell} = \lambda_{r\ell}/\lambda_{11}$, taken from reference 8. One notices, as Mott-Smith points out, that in addition to the above degeneracy, we have $\lambda_{30} = \lambda_{21} = \lambda_{02}$. This triple degeneracy is, of course, in addition to the triple degeneracy $\lambda_{00} = \lambda_{01} = \lambda_{10} = 0$. The latter, as pointed out already, is a consequence of the conservation of mass, momentum, and energy in a two particle collision. We will in the future refer to these, and the branches which they define, as the hydrodynamic roots. The reasons for this will become clear.

In order to study the character of plane wave solutions to (2.41), we assume a dependence

$$(3.4) \quad b_{r\ell} = b_{r\ell} e^{\sigma t' - ikx'}$$

for all dependent variables where the $b_{r\ell}$ on the right is a constant. This corresponds to a Laplace transform in time and Fourier transform in space. However, since we will not for the moment be interested in initial or boundary value problems, we do not formally perform the transforms. Making the substitution of (3.4) in (2.41), we get

$$(3.5) \quad (\sigma - \lambda_{\mu\nu})b_{\mu\nu} - ik \left[\nu \left(b_{\mu\nu-1} \sqrt{\frac{(\mu+\nu+1/2)^2}{(2\nu-1)(\nu\mu+1)}} \right. \right. \\ \left. \left. - b_{\mu+1,\nu-1} \sqrt{\frac{2\mu+2}{(2\nu-1)(2\nu+1)}} \right) + (\nu+1) \left(b_{\mu\nu+1} \sqrt{\frac{(\mu+\nu+3/2)^2}{(2\nu+3)(2\nu+1)}} \right. \right. \\ \left. \left. - b_{\mu-1,\nu+1} \sqrt{\frac{2\mu}{(2\nu+3)(2\nu+1)}} \right) \right] = 0$$

Using the ordering by eigenvalues we can write the system as

$$(3.6) \quad [\sigma I - ikB - \lambda]V = 0$$

where I is the identity matrix, and

$$(3.7) \quad \lambda = \begin{bmatrix} \lambda_{00} & & & & \\ & \lambda_{01} & & & \\ & & \lambda_{10} & & \\ & & & \lambda_{02} & \\ 0 & & & & - \\ & & & & - \\ & & & & - \end{bmatrix} .$$

B is the coefficient matrix of ik and V is a vector whose components are given by (2.47). The remainder of this section will be spent in locating the roots $\sigma(k)$ ¹⁹ of the determinant of equation (3.6), or equivalently finding the eigenvalues of the matrix

$$(3.8) \quad C = +ikB + \lambda .$$

The determinant of the linear homogeneous system (3.6) will be referred to as $D(\sigma, k)$. This, of course, defines the dispersion law. To avoid questions of rigor we will consider the system (3.6) as arbitrarily large but finite. By this dodge we may use the results of the function theory. A truncated matrix of the system (3.6) is given for the eigenvalue ordering in Table II and for the polynomial ordering in Table III.

19. For a boundary value problem, one is essentially interested in $k = k(\sigma)$. This case is considered in the reference of footnote 7.

On setting $k = 0$ in

$$(3.9) \quad D(k, \sigma) = 0$$

we see immediately that

$$(3.10) \quad \sigma = \lambda_{r\ell}$$

are the roots. As already noted $\sigma = 0$ and $\sigma = \lambda_{02} (= \lambda_{30} = \lambda_{21})$ are triple roots and $\sigma = \lambda_{r0} (= \lambda_{r-1,1})$ are double roots. We will now develop the roots in a power series in k in the neighborhood of each of these $k = 0$ points. Each $k = 0$ point defines the origin of one or more of the branches of (3.9). That such a power series exists follows from function theory. As is shown in Appendix 1, the dispersion relation is a function of k^2 . In the neighborhood of a simple zero the root σ , may be developed in a series of ascending powers of k^2 . However, in the neighborhood of a double or triple point a Puiseux series must be used. In our case this amounts to a power series expansion in k . For a discussion of these remarks one is referred to a book on function theory.²⁰

We write for the roots σ ,

$$(3.11) \quad \sigma = \sigma_0 + \sigma_1 ik + \sigma_2 k^2 + \dots$$

and for $D(k, \sigma)$ we make the expansion,

20. E. Goursat, A Course in Mathematical Analysis, vol. II, pt. 1, (Dover, 1959).

$$(3.12) \quad D = D_0 + k D_0^{(1)} + \frac{k^2}{2} D_0^{(2)} + \frac{k^3}{3!} D_0^{(3)} + \dots$$

The primes denote differentiation and the zero subscript the evaluation at $k = 0$. Relation (3.11) is placed in (3.12) and this results in the evaluation of the σ in (3.11), when (3.12) is placed equal to zero. The latter states that

$$(3.13) \quad 0 = D_0 = D_0^{(1)} = D_0^{(2)} = \dots = D_0^{(n)} = \dots$$

It is shown in Appendix 1 that for our particular determinant

$$(3.14) \quad \left. \frac{d^n D}{dk^n} \right|_{k=0} = \sum_{i_1 j_1 \dots i_n j_n} D(a'_{i_1 j_1}, a'_{i_2 j_2}, \dots, a'_{i_n j_n}) \Big|_{k=0}$$

As explained in the appendix, the symbol $D(a'_{ij})$ represents the determinant with its (ij) element replaced by its derivative. In general, (3.14) represents the determinant with n of its elements replaced by their respective derivatives. And when an element appears m times it represents the m^{th} derivative of that element. The relation given by (3.14) will be a great aid in the study of the dispersion relation.

In the succeeding section we will treat in general all branches which occur in the expansion of $D(\sigma, k)$. Worthy of special attention are the three branches which occur in connection with the $\sigma = 0$ triple point, the hydrodynamical case. We

treat this next and in greater detail than any other branches.

The Hydrodynamical Branches

In this case it is more convenient to examine Table III, the polynomial ordering, rather than the eigenvalue ordering of Table II which will be used in all other cases. (Of course, the results are indifferent to whatever ordering we use. We use a particular ordering only for the convenience it affords us). To facilitate remarking on Table III we have placed "abscissa" and "ordinate" lines marked 1, 2, and 3 in the table. Further, we shall refer to abscissa as i and ordinate as j . For instance $1 \leq i, j \leq 2$ refers in the table to the square

$$(3.15) \quad \begin{array}{|c|c|} \hline \sigma - \lambda_{02} & \sqrt{\frac{8}{15}} ik \\ \hline \sqrt{\frac{8}{15}} ik & \sigma - \lambda_{11} \\ \hline \end{array}$$

The hydrodynamic branches are found by taking

$$(3.16) \quad \sigma = i\alpha k + \beta k^2 + i\gamma k^3 + \delta k^4 + i\epsilon k^5 +$$

in $D(k, \sigma)$, i.e., in Table III. With the substitution of (3.16) it is clear that D , $\frac{d}{dk} D$, $\frac{d^2}{dk^2} D$ vanish identically when k is set to zero. Hence no information can be derived until we reach the third derivative. This is also clear from the fact

that $\sigma = 0$ is triple root at $k = 0$. Our formula (3.14) tells us that in order to take the third derivative of D at $k = 0$, we need only consider determinants in which three derivatives occur in any combination (i.e., three elements differentiated once, one element differentiated thrice, etc.). Unless the first three rows (columns) contain a differentiated element the row (column) vanishes, hence the third derivative at $k = 0$ at D is equivalent to the third derivative of the 3×3 determinant $i, j \leq 1$. (In this case, at $k = 0$ the remainder of the determinant is composed just of the diagonal elements which are eigenvalues λ .) In other words, for the hydrodynamical branch the dispersion relation is given by the coefficient of k^3 in the expansion of the first 3×3 determinant; we denote this by

$$(3.17) \quad \text{coef}(k^3) \begin{vmatrix} \sigma & -ik & 0 \\ -ik & \sigma & \sqrt{2/3} ik \\ 0 & \sqrt{2/3} ik & 0 \end{vmatrix} = 0$$

Computation shows that this leads to

$$(3.18) \quad \alpha = \pm \sqrt{5/3}, 0.$$

One will observe that the first two roots are, in our normalization, the two adiabatic sound speeds. More significantly, one should observe that the determinant given in (3.17) is precisely the determinant that one would get, in our normalization, if

the inviscid gas dynamic equations are Fourier and Laplace transformed. Hence from our present point of view, that of kinetic theory, the inviscid equations of fluid dynamics correctly gives the dispersion relation to $O(k)$ for sound propagation for the branches which we, in anticipation, called hydrodynamical.

The above paragraph contains the reasoning, in detail, for the determination of α . In the paragraph below we will give the reasoning in somewhat less detail, for the determination of β , i.e., the hydrodynamical dispersion correct to $O(k^4)$, and the roots to $O(k^2)$. After this just the results will be quoted and commented on, since the reasoning, although not complicated, is cumbersome to explain. It is best carried out in the privacy of one's thoughts and with the aid of (3.14) is quite straightforward.

To consider the determinant to $O(k^4)$ it is convenient to first consider the possible contribution from the elements in the square $i, j > 1$. At most one differentiated element can occur in this square (more than one gives zero identically for $k = 0$). But this element must multiply the first 3×3 determinant differentiated thrice, which vanishes by the results of the calculation to $O(k^3)$. Therefore the calculation of β is given by

$$(3.19) \quad 0 = \text{coef}(k^4) \quad \left| \begin{array}{ccccc} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & \sqrt{2/3} ik & -2/\sqrt{3} ik & 0 \\ 0 & \sqrt{2/3} ik & \sigma & 0 & -\sqrt{5/3} ik \\ 0 & -2/\sqrt{3} ik & 0 & -\lambda_{02} & 0 \\ 0 & 0 & -\sqrt{5/3} ik & 0 & -\lambda_{11} \end{array} \right|$$

For σ we must substitute each of the three possible branches,

$$(3.20) \quad \sigma = \begin{cases} 0 + \beta k^2 \\ i\sqrt{5/3} k + \beta k^2 \\ -i\sqrt{5/3} k + \beta k^2 \end{cases}$$

each of which in principle gives a different determination of β . In Appendix 1 it is shown that $D(k, \sigma)$ has only real coefficients. This shows that

$$(3.21) \quad D(k, \sigma) = D(k, \sigma^*) ,$$

i.e., if σ is a root then so also is its conjugate σ^* . Hence $\pm i\sqrt{5/3}$ lead to the same β .

It is important to note that the determinant in (3.19) is precisely the expression one would get on transforming the Navier-Stokes equations under our normalization. The last row is the transform of the relation between heat conduction and the gradient of temperature; and the fourth row the relation between stress and the space derivatives of velocity. The viscosity and heat conductivity coefficients are given respec-

tively by the reciprocals of λ_{02} and λ_{11} (to within numerical factors), in our normalization. Again from the point of view of kinetic theory, the Navier-Stokes equations give the hydrodynamical branches of the sound dispersion relation accurately to $O(k^4)$, and the roots to $O(k^2)$.

The calculation of γ is furnished by

$$(3.22) \quad 0 = \text{coef}(k^5) \left| \begin{array}{ccccc} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & \sqrt{2/3} ik & -2/\sqrt{3} ik & 0 \\ 0 & \sqrt{2/3} ik & \sigma & 0 & -\sqrt{5/3} ik \\ 0 & -2/\sqrt{3} ik & 0 & \sigma - \lambda_{02} & \sqrt{8/15} ik \\ 0 & 0 & -\sqrt{5/3} ik & \sqrt{8/15} ik & \sigma - \lambda_{11} \end{array} \right|$$

which is found in the same way as the last two determinations.

The results of the determinations indicated in (3.19) and (3.22) are,

$$(3.23) \quad \sigma = \frac{k^2}{\lambda_{11}} + O(k^4)$$

$$(3.24) \quad \sigma = \pm i \sqrt{5/3} \left[k + k^3 \left(\frac{-2}{5\lambda_{02}\lambda_{11}} + \frac{1}{10\lambda_{11}^2} + \frac{8}{15\lambda_{02}^2} \right) \right] \\ + k^2 \left(\frac{1}{3\lambda_{11}} + \frac{2}{3\lambda_{02}} \right) + O(k^4) .$$

The branch as given by (3.23) leads to a purely decaying mode and is connected to heat conduction. Relation (3.24) furnishes

the two propagating modes. With the addition of the $O(k^2)$ term the propagation acquires decay, and the $O(k^3)$ term leads to an increase in phase velocity.

The determinant of (3.22) is exactly the one which would be found from the "thirteen moments" of H. Grad², under our normalization. (Of course, in our case of one dimensional unsteady flow the number of moments are reduced to five.) Actually instead of σ appearing in the last two diagonal elements we could have put in its expansion; this was done in order to make the identification with the moments equations of Grad. As before we say that the "13 moments" equations accurately give hydrodynamical branches of the dispersion relation to $O(k^5)$. One should note that the Grad equations actually give two other branches of the dispersion relation. This will be dealt with in Section 4.

It is furthermore the case that (3.22) furnishes the same dispersion relation as would be given by the Burnett equations. This follows a procedure equivalent to the Chapman-Enskog theory, and which in our case is given by row operations on the determinant. Briefly this procedure goes in the following way. We multiply the fourth row of (3.22) by σ/λ_{02} and add it to itself; and the fifth row by σ/λ_{11} and add it to itself. The last two rows of (3.22) then becomes

$$(3.25) \quad \begin{vmatrix} 0 & \frac{-2}{\sqrt{3}} ik(1 + \frac{\sigma}{\lambda_{02}}) & 0 & -\lambda_{02} & \sqrt{\frac{8}{15}} ik \\ 0 & 0 & -\sqrt{\frac{5}{3}} ik(1 + \frac{\sigma}{\lambda_{11}}) & \sqrt{\frac{8}{15}} ik & -\lambda_{11} \end{vmatrix}$$

One notices the absence of the quadratic terms in the fourth and fifth columns. These do not appear since, as pointed out in the previous paragraph, only terms of $O(k)$ enter as the coef (k^5) in the expansion (3.22). The remainder of the reduction is carried out in the same way. The mixed terms $k\sigma$ are eliminated by multiplying the second and third rows of (3.22) by ik and suitable constants and adding to the fourth and fifth rows as given by (3.25). Again nothing new appears in the fourth and fifth columns by the previous remark. Lastly, the (4,5) and (5,4) elements are eliminated by the suitable addition of row 5 to row 4 and vice versa. The result is,

$$(3.26) \quad \begin{vmatrix} \frac{k^2}{\lambda_{02}} \sqrt{2/3} & -2/\sqrt{3} \, ik \left(\frac{-2}{3\lambda_{02}} + \frac{2\sqrt{2}}{3\lambda_{11}} \right) k^2 & -\lambda_{02} & 0 \\ 0 & \left(\frac{\sqrt{10}}{3\lambda_{11}} + \frac{4\sqrt{2}}{3\lambda_{02}} \right) k^2 - \sqrt{5/3} \, ik & 0 & -\lambda_{11} \end{vmatrix}$$

which are the transforms of the Burnett relations, in our normalization.²¹

Referring to Table III, and the determinant given by $i, j < 3$, the coefficient of k^6 leads to δ , and the coefficient of k^7 to ϵ of the expansion for σ in (3.16). The determinant is identical to the one which would be gotten from what Grad would call the "26 moments" equations. In this way we

21. C. S. Wang Chang, "On the Dispersion of Sound in Helium", APL-JHU, CM-467, UMH-3-F (1948).

may go on and accurately determine the dispersion relation of the hydrodynamic branches to any desired order. This property of being able to find σ to any order by the proper truncation of $D(\sigma, k)$ is peculiar to our determinant and is traceable to the recursion relation (2.20).

Wang Chang and Uhlenbeck⁷ also noted this property of $D(\sigma, k)$ and carried out the calculation of hydrodynamic branches to within $i, j < 3$ of Table III. They call $i, j < 1$ "Euler", $i, j < 2$ "Navier-Stokes", $i, j < 3$ "Burnett", and note that at each of these steps the dispersion relation is improved by two orders. They also note that their dispersion relation is one order more accurate than would be gotten from the equations corresponding to the appellation. We have seen in the above why this is the case. A more appropriate sequence of appellations would be "Euler", "13 moments" and "26 moments".

Before going on to the remainder of the dispersion law we pause to make some comments on the hydrodynamical branches. Although we found the transforms of the equations of the Chapman-Enskog procedure in the above analysis, it should not in any way be construed as a verification of the Chapman-Enskog procedure itself. This is clear from the fact that the inversions of the relations do not, without proper initial conditions, give Chapman-Enskog relations for stress and heat conduction. We will not deal with this in this report. Next we note that

the expansion of σ is in inverse powers of λ , i.e., with terms of the type $(k/\lambda_{r\ell})$. As we take larger determinants we gain larger $\lambda_{r\ell}$ and we can presumably describe higher frequency phenomena. Unfortunately, the calculations become more tedious with larger determinants. It should be noted that within the framework set up the expansions converge, however slowly. Again for the reasons given above, this should in no way be construed as showing convergence of the Chapman-Enskog procedure.

We now turn to the remaining branches of the spectrum and determine their behavior in the neighborhood of $\sigma = \lambda_{r\ell}$ ($k = 0$). As we will see, this will not necessarily mean that the wave number be small. It need only be small when compared with the eigenvalue $\lambda_{r\ell}$, and since this becomes increasingly large the results can be applied to larger and larger wave numbers. In no case will we carry the analysis as far as in the hydrodynamical case; the latter was given a more lengthy analysis only because of its special significance. We will separately, in turn, deal with the cases when at $k = 0$, σ is non-degenerate, doubly degenerate, and triply degenerate (the hydrodynamical case is an example of this). In referring to $D(k, \sigma)$ we shall henceforth mean Table II only.

Non-degenerate Branches

Since, as is shown in Appendix 1, $D = D(\sigma, k^2)$, we may express the branch of a non-degenerate root as

$$(3.27) \quad \sigma = \lambda_{\mu\nu} + \beta k^2 + \delta k^4 + \dots$$

With the help of the relation (3.14) it is not difficult to see that the determination of β is gotten from the coefficient of k^2 in the expansion of the determinant in which $k = 0$ except in the column and row containing $\lambda_{\mu\nu}$. This is shown in Table IV. If this is carried we get

$$(3.28) \quad \beta = - \left[(v+1)^2 \left\{ \frac{2(\mu+v+3/2)}{(2v+1)(2v+3)} \frac{1}{\lambda_{\mu\nu} - \lambda_{\mu, v+1}} + \frac{2\mu}{(2v+3)(2v+1)} \frac{1}{\lambda_{\mu\nu} - \lambda_{\mu-1, v+1}} \right\} + v^2 \left\{ \frac{2(\mu+v+1/2)}{(2v+1)(2v-1)} \frac{1}{\lambda_{\mu\nu} - \lambda_{\mu, v-1}} + \frac{2(\mu+1)}{(2v-1)(2v+1)} \frac{1}{\lambda_{\mu\nu} - \lambda_{\mu+1, v-1}} \right\} \right]$$

If one inserts into (3.24) those values which are given in Table I, it is found that $\beta_{\mu\nu}$ is negative. It is plausible to presume that the coefficient of k^2 in the non-degenerate branches are all negative. We will comment on the significance of this in Section 4. The next order may be gotten by filling in the neglected entries of Table IV and expanding the determinant.

Doubly Degenerate Branches

From (3.3) we know that the doubly degenerate eigenvalues are given by $\lambda_{r0} = \lambda_{r-1,1}$. Referring to Table II a case in

point is $\lambda_{11} = \lambda_{20}$. Using the same line of reasoning as used in the preceeding cases we see that α of $\sigma = \lambda_{11} (= \lambda_{02}) + i\alpha k + \beta k^2$ is given by the coefficient of k^2 in the expansion of determinant $1 < i, k < 2$ in Table II. Carrying out this calculation, we find

$$(3.29) \quad \alpha = \pm 2/\sqrt{3}.$$

To compute β which, as has been pointed out, is the same for both values, we need only consider the coefficient of k^3 in the determinant in which $k = 0$, except in the two columns and rows which contain $\lambda_{11}, \lambda_{20}$. On carrying out this calculation we get

$$(3.30) \quad \beta = -\frac{5}{6\lambda_{20}} - \frac{7}{6(\lambda_{20} - \lambda_{21})} - \frac{28}{30(\lambda_{20} - \lambda_{12})} - \frac{8}{30(\lambda_{20} - \lambda_{02})}.$$

The general expressions for α and β may be derived by returning to the transformed system given by equation (3.5). If the equations are reordered for the purposes of the calculation, we may write the matrix in the neighborhood of $\lambda_{r0} (= \lambda_{r-1,1})$ as given in Table V. The coefficient α is immediately seen to be

$$(3.31) \quad \alpha = \pm \sqrt{2r/3}.$$

β is determined by taking the coefficient of k^3 in the expression of the determinant of Table V. This gives

$$(3.32) \quad \beta = - \left[\frac{2r+3}{6(\lambda_{r0} - \lambda_{r,1})} + \frac{2r+1}{6(\lambda_{r0} - \lambda_{r-1,0})} + \frac{2(2r+3)}{15(\lambda_{r0} - \lambda_{r-1,2})} \right. \\ \left. + \frac{2(2r-2)}{15(\lambda_{r0} - \lambda_{r-2,2})} \right] .$$

As before, the next term in the expansion, γ , is furnished by filling in the remaining entries in Table V and expanding the resulting determinant. This is not done since the result does not seem worth the tedious labor involved. The β 's which can be computed from Table I are negative. It is again plausible to presume this negativity for all β given by (3.28).

Triply Degenerate Eigenvalues ($\lambda_{02} = \lambda_{21} = \lambda_{30}$)

We shall only state the results of this calculation since it offers nothing new in the way of computation.

$$(3.33) \quad \begin{cases} \sigma = \lambda_{30} - k^2 \left[\frac{4}{3\lambda_{02}} + \frac{8}{15(\lambda_{02} - \lambda_{11})} - \frac{9}{5(\lambda_{02} - \lambda_{03})} \right] + o(k^4) \\ \sigma = \lambda_{30} \pm \sqrt{2} ik - k^2 \left[\frac{8}{15(\lambda_{02} - \lambda_{12})} + \frac{18}{15(\lambda_{02} - \lambda_{22})} \right. \\ \left. + \frac{7}{6(\lambda_{02} - \lambda_{20})} + \frac{3}{2(\lambda_{02} - \lambda_{31})} \right] + o(k^3) \end{cases}$$

Again one may see, on using the values of Table I, that each of the modes are dissipative. No general relation is given in this case since no general triple degeneracy is to be found. In a sense this last case is not a triple degeneracy at all, at

least not in the same way as the hydrodynamical branches, is. One can easily see that the branch corresponding to λ_{02} may be computed as if it were a non-degenerate case. Further, the branches corresponding to $\lambda_{30}, \lambda_{21}$ may be computed as if they were a double degeneracy.

4. Some Remarks on Solutions of the Boltzmann Equations.

Using the data of the last section we have plotted in Figure 1 the roots of the dispersion relation in the complex σ -plane. For these curves, k occurs as a parameter and the dependence of σ on k may be found by returning to the appropriate calculation in Section 3. Our work so far has been aimed at solving an initial value problem. The purpose of this section is to avoid actually solving an initial value problem. We will do this mostly by studying Figure 1, and making some plausible assumptions. In this way we will be able to display the character of a solution to the Boltzmann equation as it evolves in time. This will all be subject to a somewhat serious restriction. Our representation of the dispersion relation was in a power series. Therefore, if only a few terms of the series are taken, the representation will presumably be valid for only small k . Hence we will be restricting ourselves to problems with sufficiently smooth initial data²² or equivalently that the observer point be far removed from the initial disturbance.

Looking at Figure 1 we see that three branches emanate from the origin. In the last section we referred to these as the hydrodynamic branches. From (3.23) we have for the branch on the real axis,

22. In our normalization, "smooth" has the meaning that the ratio of the mean free path to the wavelength of the phenomena be small.

$$(4.1) \quad \sigma = -\beta_{10}k^2, \quad \beta_{10} > 0.$$

This dispersion would also be obtained by considering the equation

$$(4.2) \quad \frac{\partial}{\partial t} u = \beta_{10} \frac{\partial^2}{\partial x^2} u$$

which we identify as the diffusion equation. Therefore the mode given by (4.1) is strictly diffusing. From (3.24) we have for the upper branch

$$(4.3) \quad \sigma = \sqrt{5/3} ik - \beta_{00} k^2 \quad \text{where} \quad \beta_{00} > 0$$

This mode therefore gives a propagation with the adiabatic speed $\sqrt{5/3}$ (in our normalization) and a dispersive part given by $\beta_{00}k^2$. The expression (4.3) could have been gotten by transforming a differential equation of the form

$$(4.4) \quad \frac{\partial u}{\partial t} + \sqrt{5/3} \frac{\partial u}{\partial x} = \beta_{00} \frac{\partial^2}{\partial x^2} u.$$

From this it is clear that one gets a "wave" propagation with the speed $\sqrt{5/3}$ around which there is a diffusion governed by β_{00} . Of course the conjugate hydrodynamic branch gives the same phenomena only propagating in the opposite direction.

Each of the other branches can be considered in the same manner. The chief difference between the other

branches and the hydrodynamical branch lies in the fact that they each start off with a negative real part. For instance, consider the second cluster of Figure 1. From the last section this is given by

$$(4.5) \quad \sigma = \lambda_{20} + \frac{2}{\sqrt{3}} ik - \beta_{20} k^2 + \dots$$

$$\beta_{20} > 0, \quad \lambda_{20} < 0.$$

Such a dispersion relation could also be gotten by transforming the following differential equation,

$$(4.6) \quad \frac{\partial u}{\partial t} + \frac{2}{\sqrt{3}} \frac{\partial}{\partial x} u - \lambda_{20} u = \beta_{20} \frac{\partial^2}{\partial x^2} u.$$

This is of the same type as (4.4). The undifferentiated term may be eliminated by taking $u = v e^{\lambda_{20} t}$, which decays in time. This would result in the same equation as (4.4). Therefore the second branch leads to a mode which propagates with a speed $2/\sqrt{3}$ while diffusing according to β_{20} , and decaying with a folding time given by λ_{20}^{-1} . We see from Figure 1, and the results of Section 3, that the branches lead to an infinite number of modes which propagate and diffuse,²³ and an infinite number which just diffuse. All but the hydrodynamical modes decay with a time faster than the collision time.

As we have mentioned in the beginning of this section,

23. Wang Chang and Uhlenbeck (ref. 7, p. 32) were aware of the possibility of more than one sound speed; they specifically point out a mode corresponding to equation (4.6).

we were not successful in calculating the eigenvalues, σ , for all k but only for small values of k . Hence our dispersion relation is only capable of dealing with problems which have relatively smooth initial data, or equivalently for arbitrary initial data the observer point must be sufficiently far from the location of the disturbance. On the other hand the dispersion law is capable of dealing with problems for all time. If one wishes to consider phenomena for times larger than the collision time the hydrodynamic branches need only be considered, since all other modes are asymptotically small. (Here, as in previous discussion, we exclude the presence of boundaries). And so for sufficiently large times and smooth initial data (or the equivalent given above) we get the same modes as given by Navier-Stokes, Burnett, etc. (see identifications made in Section 3). This again should not be construed as a derivation of these systems of equations, since we have not discussed the way in which initial data enters.

One needs not only consider the hydrodynamic branches. The modes of all the branches to the right of some arbitrary point on the real of Figure 1 may be considered, and all the modes to the left would be considered small when compared to the modes to the right. Indeed the entire solution including all the modes, may be thought of as a succession of epochs, or temporal boundary layers. Each epoch is determined by the "folding times" λ_{rl}^{-1} . After each of

these times a cluster of modes becomes, roughly speaking, e^{-1} of its initial value. This does not at all mean that it is negligible when compared to the modes immediately to the right of it in Figure 1. Inspection of Table I shows that ratios of eigenvalues need not be that large to warrant such a statement. However a particular mode can be negligible compared to modes sufficiently far enough to the right of it in Figure 1. A cluster of modes is washed out of the picture more quickly than its neighboring cluster to the right and less quickly than the one to the left. The hydrodynamic mode is the last of these epochs, and all other modes are asymptotically small compared to it.

Lastly we would like to say a few words about the infinitude of sound propagations. It is indeed the case that the infinity of sound propagations as they appear in this report are specifically a product of having chosen Maxwell molecules. In general we have no a priori reason to presume that the eigenvalue degeneracy of Maxwell molecules is likely to occur for other molecular models. However, there are several arguments which can be offered in support of the extraordinary sound speeds in general. We recall from Section 2 that from the isotropy of the Boltzmann integral operator Wang Chang and Uhlenbeck⁷ were able to show that a $(2\ell+1)$ -fold degeneracy of the eigenvalues occurs for arbitrary molecular force laws. This degeneracy was wiped out in our problem by the elimination of shearing moments.

As pointed out in Section 2, however, a general attack on one dimensional unsteady problems must include such shearing moments. Therefore a degeneracy would occur and we could expect propagation arising from it.

Still another argument in favor of the infinitude of speeds is furnished by the remarks made at the close of Section 2. For non-Maxwell molecules we no longer have the degeneracies which were responsible for the non-hydrodynamic propagations found. However from the remarks at the end of Section 2, we know that for other molecular models the trace elements of the collision matrix operator are only slightly removed from the Maxwell values. The double and triple points are replaced by adjacent sets of points. In the next section we will show how these neighboring points can lead to a degeneracy and give wave propagation. The argument will rest on the fact that each truncated system of moments equations is hyperbolic. This has the effect of producing wave propagation for sufficiently high frequency phenomena. Analytically what can happen is that the adjacent points lead to diffusion for small wave numbers but at some threshold value they coalesce and produce propagation.

5. Kinetic and Macroscopic Models

Throughout this report we have concentrated our efforts in obtaining the dispersion relation of the exact equation, i.e., for the linearized Boltzmann equation. These lead to series representations of the branches of the dispersion relation, as given by the expressions found in Section 2. Such expressions, of course, limit us to the study of relatively small wavelength phenomena. However, in principle, the series representations may be carried to any desired order, but unaided by machines, such calculations soon become discouraging. In the hope of gaining more control on the entire range of wavelengths, we now give up the notion of finding the dispersion relation for the exact equation. We shall rather explore the possibility of finding approximate formulations which lead in some sense to approximate dispersion relations.

Certainly the simplest approximation scheme would lie in using a truncated system without restriction. For instance referring to Table III, if we truncate the system $i, j < 1$ off from the infinite system we have the following. There are three roots of the dispersion relation. Two correspond to propagation in opposite directions with the adiabatic speed, $\sqrt{5/3}$, and the third is a zero speed propagation. Equivalently we could have gotten the same results by truncating the system given by the differential equations (2.41). (These can also be gotten by direct

inversion of the three by three matrix, $1, j < 1$). The propagations are of course the characteristic speeds of the Euler equations, with the zero speed corresponding to the inviscid limit of heat conduction. This system is free of dissipation and gives a propagation speed independent of the wave number. The dispersion relation is given parametrically in k , in the σ -plane in Figure 2.

The next truncation which is implied would be given by $1, j < 2$ in Table III. More directly by the corresponding truncation of the system of differential equation (2.4), we get

$$(5.1) \left\{ \begin{array}{l} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\rho + T) + \frac{\partial}{\partial x} p_{11} = 0 \\ \frac{\partial T}{\partial t} + \frac{2}{3} \frac{\partial u}{\partial x} + \frac{2}{3} \frac{\partial S_1}{\partial x} = 0 \\ -\frac{\partial p_{11}}{\partial t} + \lambda_{02} p_{11} + \frac{4}{3} \frac{\partial u}{\partial x} + \frac{8}{15} + \frac{\partial S_1}{\partial x} = 0 \\ \frac{\partial S_1}{\partial t} - \lambda_{11} S_1 + \frac{\partial p_{11}}{\partial x} + \frac{5}{2} \frac{\partial T}{\partial x} = 0 \end{array} \right.$$

These will be recognised as the thirteen moments equations of H. Grad². This was also pointed out in section 3, when

considering the hydrodynamic branches of the spectrum. In fact in the method of truncation we have been tacitly assuming that we are following the truncations demanded by the computation of the hydrodynamic branches of Section 3. A shortcoming to this method, as well as an alternative plan will be presented shortly. As has already been pointed out the "13-moments" dispersion relation contains the dispersion relations of Navier-Stokes and Burnett.

In Section 2 it was shown that any truncation no matter how large, gave a symmetric hyperbolic system of equations. This has several consequences which are of importance to us. Rather than deal with the "13-moments" equations specifically, we treat any truncated system no matter how large. First we mention that any such system has real finite characteristic speeds. In our case this means that as $k \longrightarrow \infty$ we are lead to finite speeds of propagation (some of which may vanish). Next we may estimate the dissipation of such a system. We represent a truncated system by

$$(5.2) \quad (I \frac{\partial}{\partial t} + A \frac{\partial}{\partial x} - \Lambda)b = 0$$

where I is the identity matrix, A the coefficient matrix, as gotten from (2.41), and Λ the diagonal matrix made up of the eigenvalues of the Boltzmann integral operator. The dispersion relation of (5.2) is given by,

$$(5.3) \quad \det [I\sigma - ikA - \Lambda] = 0 \quad ,$$

An equivalent problem is to find the eigenvalues σ , such that

$$(5.4) \quad (ikA + \Lambda)b_\sigma = \sigma b_\sigma$$

where b_σ is the corresponding eigenvector. Multiplying (5.4) by the conjugate b_σ , b_σ^* we get

$$(5.5) \quad ik(A b_\sigma , b_\sigma^*) + (\Lambda b_\sigma , b_\sigma^*) = \sigma(b_\sigma , b_\sigma^*)$$

Since A is real and symmetric (Ab, b^*) is real. Therefore if we denote the real part of σ by σ_r we have

$$(5.6) \quad \sigma_r = \frac{(\Lambda b_\sigma , b_\sigma^*)}{(b_\sigma , b_\sigma^*)}$$

which is a very revealing expression. Since Λ has been shown (see Section 3) to be non-positive, we have,

$$(5.7) \quad \sigma_r \leq 0 \quad .$$

Further if we denote by λ^* the eigenvalue of largest magnitude, we have further

$$(5.8) \quad \lambda^* \leq \sigma_r \leq 0 \quad .$$

Hence any truncated system has limited decay, and as the size of the truncated system increases λ^* becomes more negative. It should be born in mind that σ_r is a function of k , as is the eigenvector b_σ . The work of the previous sections indicates that σ_r becomes more negative with increasing k . A statement which is certainly physically reasonable. Further, by increasing the size of the system it is plausible to assume that $\sigma_r(k)$ will become more negative. This is clearly implied from equation (5.6). We conclude from this line of reasoning that a truncated system in limiting the decay of high wave number phenomena, inadequately describes this part of spectrum. Also in support of this statement are the calculations of section 3. We showed there that any coefficient in the series representation of σ is not correctly given until a large enough matrix is taken.

We return to the thirteen moments system as given by (5.1). The dispersion relation is given by $i, j < 2$ of Table III. For $k = 0$, the roots are $\sigma = 0, 0, 0, \lambda_{11}, \lambda_{02}$. The triple root at $(\sigma, k) = (0, 0)$ has already been analyzed for small k , in Section 3 and leads to

$$(5.9) \quad \begin{cases} \sigma = \pm \sqrt{5/3} ik + k^2 \left(\frac{1}{3\lambda_{11}} + \frac{2}{3\lambda_{02}} \right) + O(k^3) \\ \sigma = \frac{k^2}{\lambda_{11}} + O(k^4) \end{cases}$$

Using the methods of Section 3, we immediately get for the other two roots,

$$(5.10) \quad \sigma = \lambda_{11} - \frac{k^2}{\lambda_{11}} \left[5/3 + \frac{8\lambda_{11}}{15(\lambda_{11} - \lambda_{02})} \right] + O(k^4)$$

$$(5.11) \quad \sigma = \lambda_{02} - \frac{k^2}{\lambda_{02}} \left[4/3 + \frac{8\lambda_{02}}{15(\lambda_{11} - \lambda_{02})} \right] + O(k^4) .$$

On substituting the values of Table I, we find that both branches given by (5.10) and (5.11) move to the right. This is contrary to physical experience since it predicts a decrease in decay with wave number. When

$$(5.12) \quad \begin{cases} k \sim \pm .35 \\ \sigma \sim .6\lambda_{02} \end{cases}$$

these two branches come together in a double point and leave the axis in a way shown in the circles region of Figure 3. Inspection shows the disagreement with the exact dispersion relation given in Figure 1. This is directly due to the polynomial ordering chosen. We shall come back to this point after sketching in the remainder of the "thirteen moments" dispersion relation. In order to examine the dispersion relation for large k , we extract ik from each row of the determinant given by (3.22).

$$(5.13) \quad \begin{vmatrix} x & -1 & 0 & 0 & 0 \\ -1 & x & \sqrt{2}/3 & -2/\sqrt{3} & 0 \\ 0 & \sqrt{2}/3 & x & 0 & -\sqrt{5}/3 \\ 0 & -2/\sqrt{3} & 0 & x-\varepsilon & \sqrt{8}/15 \\ 0 & 0 & -\sqrt{5}/3 & \sqrt{8}/15 & x-\sigma\varepsilon \end{vmatrix} = 0$$

where

$$(5.14) \quad \begin{cases} x = \sigma/ik \\ \varepsilon = \frac{\lambda_{02}}{ik}, \quad \sigma\varepsilon = \frac{\lambda_{11}}{ik} \end{cases}$$

One then considers ε a small quantity and computes the roots in this limit. For Maxwell molecules using the values of table I, the case which is sketched, we get

$$(5.15) \quad \sigma = \pm 5/9 \lambda_{02} + O(1/k^2)$$

$$(5.16) \quad \begin{cases} \sigma = \pm ik(\sqrt{4.54}) + .33\lambda_{02} + O(1/k) \\ \sigma = \pm ik(\sqrt{.66}) + .22\lambda_{02} + O(1/k) \end{cases}$$

The coefficients of ik in (5.15) and (5.16) should be recognized as the "13 moments" characteristic speeds².

On carrying out the expansions of (5.36) to several more terms we find that the first expression is to be identified with the hydrodynamic branch and the second

with the extraneous sound. This completes the picture given in Figure 3.

We pause now to carry on our argument begun in Section 4. At that time it was stated that although a system may not have degenerate eigenvalues it would still lead to propagation. This being because the truncated systems are always hyperbolic. The mechanism by which this takes place is clear from Figure 3. If in the general case the origin of two branches are close together, we can expect them to join in the manner shown in the circle of Figure 3. Instead of moving to the right with wave number they will probably move to the left however.

We recapitulate the remarks made in connection with the thirteen moments equations. For smooth phenomena, $k \ll 1$, they accurately describe hydrodynamic branch, but not so for the branches near λ_{11} and λ_{02} . Comparison with Figure 1 shows that the dispersion relation is completely distorted in that region. The most evident inconsistency being that decay decreases with wavenumber. For large wave number phenomena the inconsistency is more striking. From what has been said, and also some remarks which occur later, the indication is that all branches of the dispersion relation move toward negative infinity with increasing wave number. The thirteen moments system, or for that matter any truncated moment system, has a finite absorption cut-off. Further aggravating this is the fact

that there is a branch for which the absorption decreases as k becomes unbounded, see Figure 3.

As we have pointed out on several occasions, the Navier-Stoke equations may be extracted from the thirteen moments equations. This may be done by direct inversion of the system corresponding to (3.19). In that case, the point of view is that one obtains a system of differential equations which gives the hydrodynamical branch accurately to $O(k^4)$. More generally (since it can be applied to non-linear systems) one can obtain it by the interpolation scheme given by Grad. (See reference 3. p. 110). In any case the system of equations is given by,

$$(5.17) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\rho + T) + \frac{4}{3\lambda_{02}} \frac{\partial^2 u}{\partial x^2} = 0 \\ \frac{\partial T}{\partial t} + \frac{2}{3} \frac{\partial u}{\partial x} + \frac{5}{3\lambda_{11}} \frac{\partial^2 T}{\partial x^2} = 0 \end{cases}$$

The dispersion relation of the Navier-Stokes systems is of course given by (3.19). It contains only the hydrodynamical branch, which it gives faithfully for small wave number. The dispersion relation sketched for all wave number is given in Figure 4.

The behavior near the origin is given to $O(k^3)$ by (3.23) and (3.24). A simple calculation shows that for

large k , the three roots are,

$$(5.18) \quad \sigma = \frac{5\lambda_{11}}{5\lambda_{02} + 4\lambda_{11}} + O(1/k^2)$$

$$(5.19) \quad \sigma = \frac{4k^2}{3\lambda_{02}} + O(1)$$

$$(5.20) \quad \sigma = \frac{5k^2}{3\lambda_{11}} + O(1)$$

So, as is sketched in Figure 4, the two propagating modes return to the real axis in a double point and proceed independently along the real axis. This behavior could also have been noticed earlier from equations (5.17). There, if $k \longrightarrow \infty$ (or equivalently $\frac{\partial}{\partial x} \longrightarrow \infty$) the single space derivatives in the last two equations are ignorable with respect to the dissipative derivatives. And the phenomena is essentially governed by two diffusion equations.

This state of affairs is somewhat ironic. The Navier-Stokes equations have been extracted from the thirteen moments equations. On comparison of Figure 3 and 4, the Navier Stokes equations give a bad approximation to the latter equations for high frequency phenomena. They, in contrast to the moments equation, give unbounded absorption with increasing wavenumber. On the other hand the moments equations give a bad approximation to the Boltzmann equation for large k . The irony of course is that the Navier-Stokes equations qualitatively at least approximate

the Boltzmann equation, for large k .

The remarks made thus far indicate how the thirteen moments might be improved so as to give a more faithful approximation to the exact equations. First it is obvious that the relative placing of the eigenvalues must be given consideration. From relaxation considerations it is clear that in a homogeneous problem, the moment b_{20} persists for a longer time than does the stress b_{02} (since it is represented by a smaller eigenvalue). Looking at Table II the eigenvalue ordering, it is tempting to consider the truncated system $i, j < 2$. This for small k would give the first two clusters of branches in Figure 1. The immediate shortcoming to this system is that it excludes the stress. This would have the consequence of giving the hydrodynamic branches incorrectly to $O(k^2)$. Alternately the Navier-Stokes could not be extracted from this system. A more plausible system which rectifies this difficulty, is given by the first four equations of (5.1) and the following two,

$$(5.21) \quad \frac{\partial S_1}{\partial t} - \lambda_{11} S_1 + \frac{\partial p_{11}}{\partial x} + \frac{5}{2} \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} Q = 0$$

$$(5.22) \quad \frac{\partial}{\partial t} Q - \lambda_{20} Q + \frac{4}{3} \frac{\partial}{\partial x} S_1 = 0$$

where

$$(5.23) \quad Q = \sqrt{10/3} \, b_{20} = \frac{1}{(2\pi)^{3/2}} \int e^{-\xi^2/2} g \left[\frac{\xi^4}{2} - 5\xi^2 + \frac{15}{2} \right]$$

The system given by (5.21), (5.22) and the first four equations of (5.1), still shares with all truncated systems the shortcoming of having bounded absorption. One way to overcome this is to use the interpolation scheme suggested in the extraction of the Navier-Stokes from the thirteen moments equations. In place of the fourth equation of (5.1) we take instead

$$(5.24) \quad p_{11} = - \frac{4}{3\lambda_{02}} \frac{\partial u}{\partial x} - \frac{8}{15\lambda_{02}} \frac{\partial S_1}{\partial x}$$

Combining, the entire system reads,

$$(5.25) \quad \left\{ \begin{array}{l} \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\rho + T) - \frac{4}{3\lambda_{02}} \frac{\partial^2 u}{\partial x^2} - \frac{8}{15\lambda_{02}} \frac{\partial^2 S_1}{\partial x^2} = 0 \\ \frac{\partial T}{\partial t} + \frac{2}{3} \frac{\partial u}{\partial x} + \frac{2}{3} \frac{\partial S_1}{\partial x} = 0 \\ \frac{\partial S_1}{\partial t} - \lambda_{11} S_1 - \frac{4}{3\lambda_{02}} \frac{\partial^2 u}{\partial x^2} - \frac{8}{14\lambda_{02}} \frac{\partial^2 S_1}{\partial x^2} + \frac{5}{3} \frac{\partial T}{\partial x} + \frac{\partial Q}{\partial x} = 0 \\ \frac{\partial Q}{\partial t} - \lambda_{20} Q + \frac{4}{3} \frac{\partial}{\partial x} S_1 = 0 \end{array} \right.$$

This system is easily generalizable to a non-linear system.

We will not study this nor will we study (5.25), in this report.

We are interested in the more general approximations to the Boltzmann equations which will be made shortly.

Implicit in each of the truncation and associated interpolation methods is a particular form of the distribution function. Equivalent to the method of straightforward truncation is the choice of a distribution function of the form,

$$(5.26) \quad g = \sum_{r=0}^N b_r \psi_r$$

where some particular ordering is taken. This is substituted into the linearized Boltzmann equation, and moments are taken. One then finds that the first $(N+1)^{24}$ equations form a determined system for the coefficients b_r ($r = 0, 1, \dots, N$) which is identical to the truncated system. After solving this system of equations for the b 's, they are substituted into (5.26) above, and the corresponding distribution function is known. This is in fact the general procedure used by H. Grad². In order to obtain the interpolation scheme, one takes the interpolation equations and substitute them into equation (5.26). We illustrate this procedure for both the Navier-Stokes equations and the thirteen moments equations.

24. Actually more than $(N+1)$ equations may be generated. The embarrassing situation of having more equations than unknown is obtained. This can be avoided by ignoring equations without time derivatives.

In order to obtain the thirteen moment equations, one takes, with H. Grad², for the distribution function,

$$(5.27) \quad g = \hat{\rho} + \hat{u}_1 \hat{\xi}_1 + \frac{\hat{T}}{2} \left(\frac{\hat{\xi}^2}{3} - 1 \right) \\ + \frac{\hat{p}_{11}}{4} \left(\xi_1^2 - \frac{\hat{\xi}^2}{3} \right) + \hat{S}_1 \hat{\xi}_1 \left(\frac{\hat{\xi}^2}{5} - 1 \right)$$

This is substituted into the linearized equation, moments are taken and the system (5.1) is obtained. On solving the latter system one substitutes the solution into (5.26) and the distribution in this approximation is known. As pointed out earlier in this section, in the Navier Stokes interpolation one uses,

$$(5.28) \quad \hat{p}_{11} = \frac{4}{3\lambda_{02}} \frac{\partial u}{\partial x}$$

$$(5.29) \quad \hat{S}_1 = \frac{5}{2\lambda_{11}} \frac{\partial T}{\partial x} \quad .$$

These latter two equations are substituted into (5.26), to get

$$(5.30) \quad g = \hat{\rho} + \hat{u}_1 \hat{\xi}_1 + \frac{\hat{T}}{2} \left(\frac{\hat{\xi}^2}{3} - 1 \right) + \frac{1}{\lambda_{02}} \frac{\partial u}{\partial x} \\ + \left(\xi_1^2 - \frac{\hat{\xi}^2}{3} \right) \frac{1}{\lambda_{11}} \frac{\partial T}{\partial x} \hat{\xi}_1 \left(\frac{\hat{\xi}^2}{2} - \frac{5}{2} \right)$$

This is substituted into the linearized Boltzmann equation. moments are taken and the Navier Stokes equations result. The

latter procedure is analogous to the Chapman-Enskog procedure.

The remarks made about the truncated systems now apply to the corresponding distribution functions. Essentially that they can only be presumed to be asymptotically valid. One further remark in support of this is the following. In demanding that a distribution be of the form (5.27) or (5.30) we eliminate consideration of free flow or near free flow, for the insertion of such a distribution function in a free flow operator immediately leads to the generation of the higher moments which have been excluded from consideration.

We now investigate another type of approximation to the Boltzmann equation which is: (1) asymptotically correct ; (2) does take into consideration the higher moments ; (3) and gives a transition to free flow.

We begin by considering a relaxation theory, i. e. the space homogeneous case. Setting the space derivatives of (2.41) to zero we get

$$(5.31) \quad \left(\frac{\partial}{\partial t} - \lambda_{\mu\nu} \right) b_{\mu\nu} = 0$$

Since all but the first three λ 's (which are zero) are negative the moments $b_{\mu\nu}$ vanish exponentially. Since $|\lambda_{\mu\nu}| \rightarrow \infty$ the higher moments vanish more rapidly. It seems plausible therefore if, in the space inhomogeneous case we choose all $\lambda_{\mu\nu}$, after some point to be a single constant, that we will

overestimate the effect of the higher moment. The relaxation theory would then be correct for $b_{\mu\nu}$'s which occur before $b_{\mu,\nu}$. For those which come after, it is qualitatively correct, giving exponential decay, but with the incorrect decay time.

We carry out this operation in connection with equation (2.36). Instead of the latter equation we write,

$$(5.32) \quad \left(\frac{\partial}{\partial t} + \hat{\xi}_1 \frac{\partial}{\partial x} \right) \sum_{r,l} b_{rl} \psi_{rl} = \sum_{0,0}^{p,q} b_{rl} \psi_{rl} \lambda_{rl} + \lambda \sum_{r,l > p,q}^{\infty} b_{rl} \psi_{rl}$$

The remarks we are now making are for the polynomial order, and under this manipulation, Table III changes only in the diagonal. After some points on the diagonal the $\lambda_{r,l}$'s are replaced by λ , which is still to be specified. We take λ to be the next eigenvalue so as to accurately give the equation for one more moment. The advantage to equation (5.32) lies in a form into which it may be transformed. Adding "zero" to the right hand side of (5.32), we have,

$$(5.33) \quad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} \right) g = \sum_{\infty}^{pq} b_{rl} \psi_{rl} \lambda_{rl} - \lambda \sum_{\infty}^{p,q} b_{rl} \psi_{rl} + \lambda \sum_{0,0}^{\infty} b_{rl} \psi_{rl} = \lambda g + \sum_{\infty}^{p,q} b_{rl} (\lambda_{rl} - \lambda) \psi_{rl}$$

Equation (5.33) is precisely the equation developed by Gross and Jackson⁶ as a generalization of the equation found in the paper by Bhatnager, Gross, and Krook⁴. As an example of (5.33), we take $\lambda = \lambda_{02}$ and $(p,q) = (1,0)$, then we get

$$(5.34) \quad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} - \lambda_{02} \right) g = -\lambda_{02} \left[\hat{\rho} + \hat{u}_1 \hat{\xi}_1 + \frac{3}{2} \hat{T} \left(\frac{\hat{\xi}^2}{5} - 1 \right) \right]$$

This is, the linearized form of the single relaxation model found in references 4 and 6,

$$(5.35) \quad \left(\frac{\partial f}{\partial t} + \xi_1 \frac{\partial}{\partial x} \right) f = -\lambda_{02} (f_o - f)$$

where f_o is the local maxwellian,

$$(5.36) \quad f_o = \frac{\rho(x,t)}{[2\pi RT(x)]^{3/2}} e^{\frac{-[\xi - u(x)]^2}{2RT(x)}}$$

The linearization is that given by (2.4) and $-\lambda_{02}$ is the "collision frequency". Another model worth noting is gotten by taking $\lambda = \lambda_{03}$ and $(p,q) = (1,1)$, then

$$(5.37) \quad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} - \lambda_{03} \right) g = -\lambda_{03} \left[\hat{\rho} + \hat{\xi}_1 \hat{u}_1 + \frac{3}{2} \hat{T} \left(\frac{\hat{\xi}^2}{5} - 1 \right) \right. \\ \left. \left(1 - \frac{\lambda_{02}}{\lambda_{03}} \right) \frac{3}{4} \hat{p}_{11} \left(\frac{\hat{\xi}_1^2}{5} - \frac{\hat{\xi}^2}{5} \right) + \left(1 - \frac{\lambda_{11}}{\lambda_{03}} \right) \hat{s}_1 \hat{\xi}_1 \left(\frac{\hat{\xi}^2}{5} - 1 \right) \right]$$

Appropriate names for (5.35) and (5.37) might be the Euler and thirteen moments models respectively. The first model (5.34), for the problem of sound propagation, was studied in reference (4). A much more detailed analysis of sound propagation using this model has been performed by the author and J. Thurber²⁵. In that paper (5.37) is also applied to sound propagation.

We recall that these models (5.35) and (5.37), were obtained from Table III by replacing all λ 's past a certain point by a single constant. Let us consider the hydrodynamic branch of these models. First we consider equation (5.35). Referring to Table III, we see that $i, j < 1$ remains unchanged and the three branches in the hydrodynamic cluster is given correctly to $O(k)$. To the next order however we see that λ_{11} is replaced by λ_{02} and the expansions for λ_{02} and σ are somewhat modified. In fact all higher orders are modified in this way. However the complexion of the matrix of Table III has not been changed, and the series given by the single relaxation model is of the correct form but with incorrect constants. A truncation theory for this case may be developed as in the case of the

-
- 25. L. Sirovich and J. Thurber, "Sound Propagation according to Kinetic Models". NYO-9757, Institute of Mathematical Sciences, New York University (August, 1961).
 - 26. There is some arbitrariness at this point. We have chosen λ in (5.32) so that the propagating branches are correct to $O(k^2)$. In this case the diffusing mode is given inaccurately. Equivalently we could give the diffusion and inaccurately give the propagation. The pertinent point is that we have just one parameter left open to us. We can correctly specify the viscosity of the heat conductivity, but not both in general.

exact equation. The lowest truncation $i, j < 1$ gives, of course, the Euler dispersion relation of Figure 1. This is also indicated in Figure 5, where the effect of successively higher truncations on the hydrodynamic branches are sketched by the dotted lines moving to the left. The diffusion mode moves to the left along the real axis, as is shown. The exact theory of the single relaxation model is given by the heavy lines terminating in $\text{Re} \sigma = \lambda_{02}$ ²⁷. From the remarks made earlier, we know that no truncated system can have a greater absorption than that given by λ_{02} . For this reason that line $\text{Re} \sigma = \lambda_{02}$ plays the role it does in Figure 5. Nothing has been said of the infinity of branches which have their origin at $\sigma = \lambda_{02}$. For any finite truncation these extraordinary branches appear in the strip $\lambda_{02} \leq \text{Re} \sigma \leq 0$. As the order of the truncation increases these are shoved over to line, $\text{Re} \sigma = \lambda_{02}$, and in fact the latter line belongs to the spectrum.

It is also revealing to consider the systems of partial differential equations gotten by truncation and interpolation, as was done for the exact Boltzmann equation. As was indicated the Euler equations drop out as the first truncation. The thirteen moments come out as the second truncation, however with only the one constant. For the first interpolation the Navier-Stokes result, however again with just one constant.

27. This line, $\text{Re} \sigma = \lambda_{02}$, is also included in the spectrum, see reference 25.

One may choose it to agree either with the viscosity or the heat conductivity. This has been already remarked upon²⁸. One gets the higher moment equation and the higher interpolation correctly, but with the remark about one constant always applying.

We can comment in the same way about equation (5.37). Again referring to Table III, all λ 's after λ_{03} are set equal to λ_{03} . Recalling the calculations made in connection with the exact theory we see that the Euler, Thirteen Moments, Navier-Stokes, and Burnett equations of equation (5.37) are all identical with the similar forms gotten from the exact Boltzmann equation. The comparable statement can be made in reference to the dispersion relation. The hydrodynamical branches are given correctly to $O(k^5)$. A sketch of the dispersion relation is given in Figure 6. One can identify the first set of dotted lined with the thirteen moments dispersion law given in Figure 3. Successively higher approximations are given by the dotted lines which go to the left. The heavy dark line is the exact spectrum sketched from the results in reference (25). Again the peculiar effect of the circled region in Figure 3, is exhibited. The effect of taking more moments is to move this double point to the left. The same sort of remarks can be made now, as in the previous case, and again the reader is referred to reference (25) for a detailed investigation.

28. L. Sirovich "On the Kinetic Theory of Steady Gas Flows" NYO-9086, Institute of Mathematical Sciences, New York University (1960).

Once again the shortcomings of choosing a polynomial ordering occur in connection with the branches starting at λ_{11} and λ_{02} . We now offer a method alternate to the one suggested by Gross and Jackson⁶.

The discussion parallels that used in modifying the thirteen moments equations. From Figure 1 we see that equation (5.37) does not represent the exact dispersion law in the neighborhood of λ_{11} , λ_{02} . In order to rectify this we choose the ordering implied by the eigenvalues according to Table II. Instead of equation (5.37) we want to include the moment Q as defined by equation (5.23). Mimicking the arguments used in developing the equations given by (5.21), (5.22), we take our model equation,

$$(5.38) \quad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} - \lambda_{02} \right) g = -\lambda_{02} \left[\hat{\rho} + \hat{\xi}_1 \hat{u}_1 + \frac{3}{2} \hat{T} \left(\frac{\xi^2}{5} - 1 \right) \right. \\ \left. \left(1 - \frac{\lambda_{11}}{\lambda_{02}} \right) \hat{S}_1 \hat{\xi}_1 \left(\frac{\xi^2}{5} - 1 \right) + \left(1 - \frac{\lambda_{20}}{\lambda_{02}} \right) Q \left(\frac{\xi^4}{2} - 5\xi^2 + \frac{15}{2} \right) \right]$$

Referring to Table II we can make the following statements about equation (5.38). On setting the λ 's from λ_{02} onward equal to λ_{02} we only change the diagonal for $i, j > 3$. That is, the relaxation times of eight moments ($ij < 3$) is given faithfully. This is to be compared with (5.37) which gives six relaxation times correctly. Furthermore the hydrodynamic branches are given correctly to $O(k^3)$ with the accuracy as given by (5.37). On the other hand the second

branch is given to $O(k)$ by (5.38), but not at all by (5.37), as already mentioned. In connection with the truncated and interpolated equations, we can obtain the Euler, Navier-Stokes, Burnett and thirteen moments systems from (5.38), all which are also obtainable from (5.37). However (5.38) also furnishes the system given in connection with (5.21), (5.22) and also the associated interpolated system given by (5.25). Although the dispersion relation associated with (5.38) has not been studied yet, it is reasonable to conclude that it will have a dispersion relation of the form sketched in Figure 7.

It is interesting to note that both (5.37) and (5.38) have the same number of moments on the right hand side. However in order to rectify the same difficulty in connection with the thirteen moments equations it was necessary to introduce another moment Q . This simplification is due to the device of Gross and Jackson⁶, used for equation (5.35). This allows us to include accurately at least one more moment than is given explicitly on the right hand side of the model equation.

It should not be construed, that for purposes of truncation, interpolation or constructing models, that the eigenvalue ordering is any more suitable than the polynomial ordering which is generally used. For instance, in almost any conceivable problem in fluid mechanics the hydrodynamic mode is all one is interested in. As we saw in Section 3,

the polynomial ordering is the most natural for determining the hydrodynamic branches. An entirely different ordering (neither eigenvalue or polynomial) would be used if one wanted to examine another branch. Examples of this are given by Tables IV and V. In any problem for which either a model equation or a truncated system is to be used, one must decide how much of each branch to describe exactly. Going back to our discussion of the thirteen moments we can make this more concrete. There we saw that the thirteen moments truncation is the one necessary for the exact description of the hydrodynamic branches to $O(k^3)$. In the process however the circled area in Figure 3 appears. This seriously distorts the same area in the exact picture given by Figure 1. There were then two immediate courses open to us if we wished to rectify this situation. We could drop the stress and stress equation and replace it by Q and the associated equation. This introduced no more complexity in as much the number of equations remained equal to five. However it reduced the accuracy of the hydrodynamic branches to $O(k^2)$. If by prearranged agreement, no more than five equations are allowed, and accuracy in the hydrodynamic region is required then only the thirteen moments system must be taken. We must then be content with a poor description of the circled region of Figure 3. It seems a moot point as to whether a poor description is better than no description at all. In the latter case we might as well discard the thirteen moments

system in favor of an interpolation such as Navier-Stokes or Burnett. If on the other hand we allow ourselves six equations, we can use the system suggested earlier and connected with equation (5.22) or its associated interpolation (5.25). Each is gotten from the eigenvalue ordering of Table II (each of these gives the second cluster to $O(k)$). The hydrodynamic roots in the next truncation are gotten to $O(k^5)$ which is the system of eight equations comparable to the 8×8 truncation of Table III.

In general there is no systematic way in which to order the eigenfunctions. This can be more clearly pointed out if we allow ourselves the luxury of a large number of moments in our description. Suppose for example we choose to use the polynomial ordering. We would have then the hydrodynamical branch given quite accurately but we would have also the incorrect evolution of a great number of moments. Many of the moments which occur between those involved in the hydrodynamical calculation will not even appear. On the other hand strict use of the eigenvalue ordering leads to a correct dispersion for small k , but does not necessarily give any branch to a great accuracy. A compromise between the two methods might be advantageous. One chooses which branches to describe with some degree of accuracy and then collects the moments accordingly. This of course corresponds to the neither the eigenvalue nor polynomial ordering.

We close by considering the model equations from a formal

point of view. The following remarks are independent of the ordering chosen. This being the case we represent a model equation by,

$$(5.39) \quad \left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + v \right) g = \sum_{n=0}^N b_n v_n \psi_n$$

The identification with the previous equations are easily made, v_n being related to $(1 - \frac{\lambda_{r\ell}}{\lambda N})$, and ψ_n being the eigenfunctions $\psi_{r\ell}$. If we regard the coefficients b_n as known functions of (x, t) , equation (5.39) can be easily integrated. The result is

$$(5.40) \quad g(x, t, \xi) = e^{-vt} \int_0^t e^{vs} \sum_{n=0}^N b_n(x + \xi_1 \{s - t\}, s) \psi_n v_n ds \\ + g(x - \xi_1 t, \xi, 0) e^{-vt}$$

the last term being the initial data. As the integrated equation indicates the integration is along particle paths. We can now operate on (5.40) with

$$(5.41) \quad \int \psi_n f^0 d\xi \quad \text{for} \quad n = 0, 1, \dots, N$$

to get a system of linear equations in the $N + 1$ variables $b_n(x, t)$, $n = 0, 1, \dots, N$ ²⁹.

29. This may be reduced to a system of Abel integral equations.

On solution of the integral equations, the quantities are substituted into (5.40) and the perturbed distribution function is obtained. We illustrate the procedure for $N = 2$, the single relaxation model (5.34). Letting $\nu = -\lambda_{11}$, the solution given by (5.40) is

$$(5.42) \quad g(x, t, \xi) = e^{-\nu t} \int_0^t e^{\nu s} [\rho(x + \xi_1 \{s - t\}, s) + \xi_1 \hat{u}(x + \xi_1 \{s - t\}, s) + T(x + \xi_1 \{s - t\}, s) (\frac{\xi^2}{2} - \beta)] ds + g(x - \xi_1 t, \xi, 0) e^{-\nu t}.$$

Then operating on (5.42) with (5.41) for $n = 0, 1, 2$, we get the system

$$(5.43) \quad \begin{bmatrix} \rho(x, t) \\ u(x, t) \\ T(x, t) \end{bmatrix} = e^{-\nu t} \int \begin{bmatrix} 1 \\ \xi_1 \\ (\frac{\xi^2}{2} - \beta) \end{bmatrix} g(x - \xi, t, \xi_1, 0) f^0 d\xi + e^{-\nu t} \int_0^t e^{\nu s} f^0 \begin{bmatrix} 1 & \xi_1 & (\frac{\xi^2}{2} - \beta) \\ \xi_1 & \xi_1^2 & \xi_1 (\frac{\xi^2}{2} - \beta) \\ (\frac{\xi^2}{2} - \beta) & \xi_1 (\frac{\xi^2}{2} - \beta) & (\frac{\xi^2}{2} - \beta)^2 \end{bmatrix} \begin{bmatrix} \rho(s) \\ u(s) \\ T(s) \end{bmatrix} ds,$$

where the argument of the elements in the last vector is $(x + \xi_1 \{s - t\}, s)$. For instance $\rho(s) = \rho(x + \xi_1 \{s - t\}, s)$.

One solution of this system, the flow (ρ, u, T) is substituted into (5.42) giving the perturbed distribution function. We see that the distribution function is given in terms of the initial distribution and the evolution of the flow (ρ, u, T) . In general any model gives the distribution function in terms of the initial distribution and the evolution of a finite number of moments. In a certain sense this is a generalization of the moments method of Grad. There, as we saw earlier, (5.27), the distribution is given in terms of a finite number of moments and one solves differential equations for the moments. However full use of the initial data is not made there, and as our considerations indicated, the distribution function is only asymptotically valid. On the other hand the model solution describes the flow as we approach free flow. In fact on setting the v 's to zero in (5.40) we get the free flow solution.

APPENDIX I: Two Lemmas on Determinants

Notation

Let

$$(I.1) \quad D^0 = D(a_{\mu\nu})$$

be an $n \times n$ determinant with elements $a_{\mu\nu}$ which are differentiable functions. We write

$$(I.2) \quad \frac{da_{\mu\nu}}{dx} = a'_{\mu\nu}.$$

By

$$D(a'_{\mu\nu})$$

we shall mean the determinant with the $\mu\nu$ element replaced by its derivative. Further we write

$$(I.3) \quad D^1 = \sum_{\mu\nu}^n D(a'_{\mu\nu})$$

where the summation is over all elements. In general,

$$(I.4) \quad D^k = \sum_{\mu_1\nu_1, \dots, \mu_k\nu_k}^n D(a'_{\mu_1\nu_1}, \dots, a'_{\mu_k\nu_k})$$

where the summation is over all values of each pair of subscripts. If an element appears m ($m \leq k$) times, it denotes the m^{th} derivative. With this notation our first lemma is;

Lemma 1.

$$(I.5) \quad \frac{d^m}{dx^m} D = (D^1 - n(n-1))^m D^0 .$$

The right hand side of (I.5) represents the formal binomial expansion, with D^k given by (I.4).

To prove this we start with the well-known expansion for the derivative of a determinant

$$\frac{d}{dx} D = \sum_{\mu=1}^n D(a'_{\mu 1}, a'_{\mu 2}, \dots, a'_{\mu n})$$

but

$$D(a'_{\mu 1}, a'_{\mu 2}, \dots, a'_{\mu n}) + (n-1)D^0 = \sum_{\nu=1}^n D(a'_{\mu \nu})$$

and on summing this over all rows we get

$$(I.6) \quad \frac{d}{dx} D = D^{(1)} = D^1 - n(n-1)D^0 = [D^1 - (n-1)n]D^0 .$$

Differentiating (I.6) and applying it to itself we see that (I.5) is true for $m=2$. Using the inductive hypothesis, we have

$$\begin{aligned} \frac{d^n D}{dx^n} &= D^{(n)} = \frac{d^{n-1}}{dx^{n-1}} D^1 = [D^1 - n(n-1)]^{n-1} D^1 \\ &= [D^1 - n(n-1)]^{n-1} (D^1 - n(n-1))D^0 \\ &= [D^1 - n(n-1)]^n D^0 \end{aligned}$$

and the lemma is proven.

In our work we are interested in evaluating the determinant $D(\sigma, k)$ and its derivatives at $k=0$. Further, according to equation (3.13) we get

$$(I.7) \quad 0 = D^0 = D^1 = D^2 = \dots = D^n .$$

Our second lemma gives a sufficient condition for a determinant to yield an equation with real coefficients. We consider determinants whose elements are either real or imaginary. A zero element is considered as having both parities.

Lemma 2. A determinant for which the parities of a row may be gotten by multiplying the preceeding row by i ; or any determinant which may be put into this form (say by extracting i 's from rows) has one parity. That is, the expansion of the determinant results in a form all of whose coefficients have the same parity.

The lemma is clearly true for a two by two determinant. Expanding an $n \times n$ determinant (which staisfies the hypothesis) along the first column we get n $(n-1)$ $(n-1)$ determinants. The first determinant of the expansion satisfies the hypothesis, the second determinant can be given the same form by interchanging the first and second rows. In fact each determinant of the expansion can be put into the form of the first determinant by interchanging rows and extracting an i when necessary. Imposing the inductive hypothesis we prove the lemma.

$r \rightarrow$	0	1	2	3	4	5
0	0	0	3/2	9/4	2.808	3.274
1	0	1	7/4	2.354	2.864	3.318
2	1	3/2	2.014	2.500	2.952	
3	3/2	1.8420	2.238	2.646	3.064	
4	1.8420	2.106	2.428	2.780		
5	2.106	2.320	2.598			
6	2.320					

TABLE I
Normalized Eigenvalues $\frac{\lambda_{rl}}{\lambda_{11}}$

(The values of the table were obtained from Reference 8.)

2

TABLE II
Eigenvalue Ordering

$b_{00} = \rho$	$b_{01} = u_1$	$b_{10} = \sqrt{3}/2 \hat{t}$	b_{02}	b_{11}	b_{03}	b_{20}	b_{12}	b_{04}	b_{21}	b_{13}
σ	$-1k$									
$-1k$	σ	$/\sqrt{2/3} \ 1k$	$-2//\sqrt{3} \ 1k$							
	$/\sqrt{2/3} \ 1k$	σ		$-/\sqrt{5/3} \ 1k$						
	$-2//\sqrt{3} \ 1k$		$\sigma-\lambda_{02}$	$/\sqrt{8/15} \ 1k$	$-3//\sqrt{5} \ 1k$					
		$-/\sqrt{5/3} \ 1k$	$/\sqrt{8/15} \ 1k$	$\sigma-\lambda_{11}$		$2//\sqrt{3} \ 1k$	$-/\sqrt{28/15} \ 1k$			
			$-3//\sqrt{5} \ 1k$		$\sigma-\lambda_{03}$		$/\sqrt{18/35} \ 1k$	$-4//\sqrt{7} \ 1k$		
				$2//\sqrt{3} \ 1k$		$\sigma-\lambda_{20}$			$-/\sqrt{7/3} \ 1k$	
				$-/\sqrt{28/15} \ 1k$	$/\sqrt{18/35} \ 1k$		$\sigma-\lambda_{12}$		$/\sqrt{16/15} \ 1k$	$-/\sqrt{51/35} \ 1k$
					$-4//\sqrt{7} \ 1k$			$\sigma-\lambda_{04}$		

TABLE III
Polynomial Ordering

(μ, ν)	$(\mu, \nu-1)$	$(\mu+1, \nu+1)$	$(\mu, \nu+1)$	$(\mu-1, \nu+1)$
$\sigma^{-\lambda}_{\mu\nu}$	$-ik\nu \sqrt{\frac{(\mu+\nu+1/2)2}{(2\nu-1)(2\nu+1)}}$	$ik\nu \sqrt{\frac{2\mu+2}{(2\nu-1)(2\nu+1)}}$	$-ik(\nu+1) \sqrt{\frac{(\mu+\nu+3/2)2}{(2\nu+3)(2\nu+1)}}$	$ik(\nu+1) \sqrt{\frac{2\mu}{(2\nu+3)(2\nu+1)}}$
$-ik\nu \sqrt{\frac{(\mu+\nu+1/2)2}{(2\nu-1)(2\nu+1)}}$	$\sigma^{-\lambda}_{\mu, \nu-1}$			
$ik\nu \sqrt{\frac{2\mu+2}{(2\nu+1)(2\nu-1)}}$		$\sigma^{-\lambda}_{\mu+1, \nu+1}$.
$-ik(\nu+1) \sqrt{\frac{(\mu+\nu+3/2)2}{(2\nu+3)(2\nu+1)}}$			$\sigma^{-\lambda}_{\mu, \nu+1}$	
$ik(\nu+1) \sqrt{\frac{2\mu}{(2\nu+3)(2\nu+1)}}$				$\sigma^{-\lambda}_{\mu-1, \nu+1}$

TABLE IV
Matrix for the Diffusing Mode Calculation

$(r,1)$	$(r-1,1)$	$(r,0)$	$(r-1,0)$	$(r-1,2)$	$(r-2,2)$
$\sigma^{-\lambda}_{r,1}$		$-ik\sqrt{\frac{2r+3}{3}}$			
	$\sigma^{-\lambda}_{r,1,1}$	$ik\sqrt{\frac{2r}{3}}$	$-ik\sqrt{\frac{2r+1}{3}}$	$-ik2\sqrt{\frac{2r+3}{15}}$	$ik2\sqrt{\frac{2r-2}{15}}$
$-ik\sqrt{\frac{2r+3}{3}}$	$ik\sqrt{\frac{2r}{3}}$	$\sigma^{-\lambda}_{r0}$			
	$-ik\sqrt{\frac{2r+1}{3}}$		$\sigma^{-\lambda}_{r-1,0}$		
	$-ik2\sqrt{\frac{2r+3}{15}}$			$\sigma^{-\lambda}_{r-1,2}$	
	$ik2\sqrt{\frac{2r-2}{15}}$				$\sigma^{-\lambda}_{r-2,2}$

TABLE V
Matrix for the Propagating Mode Calculation

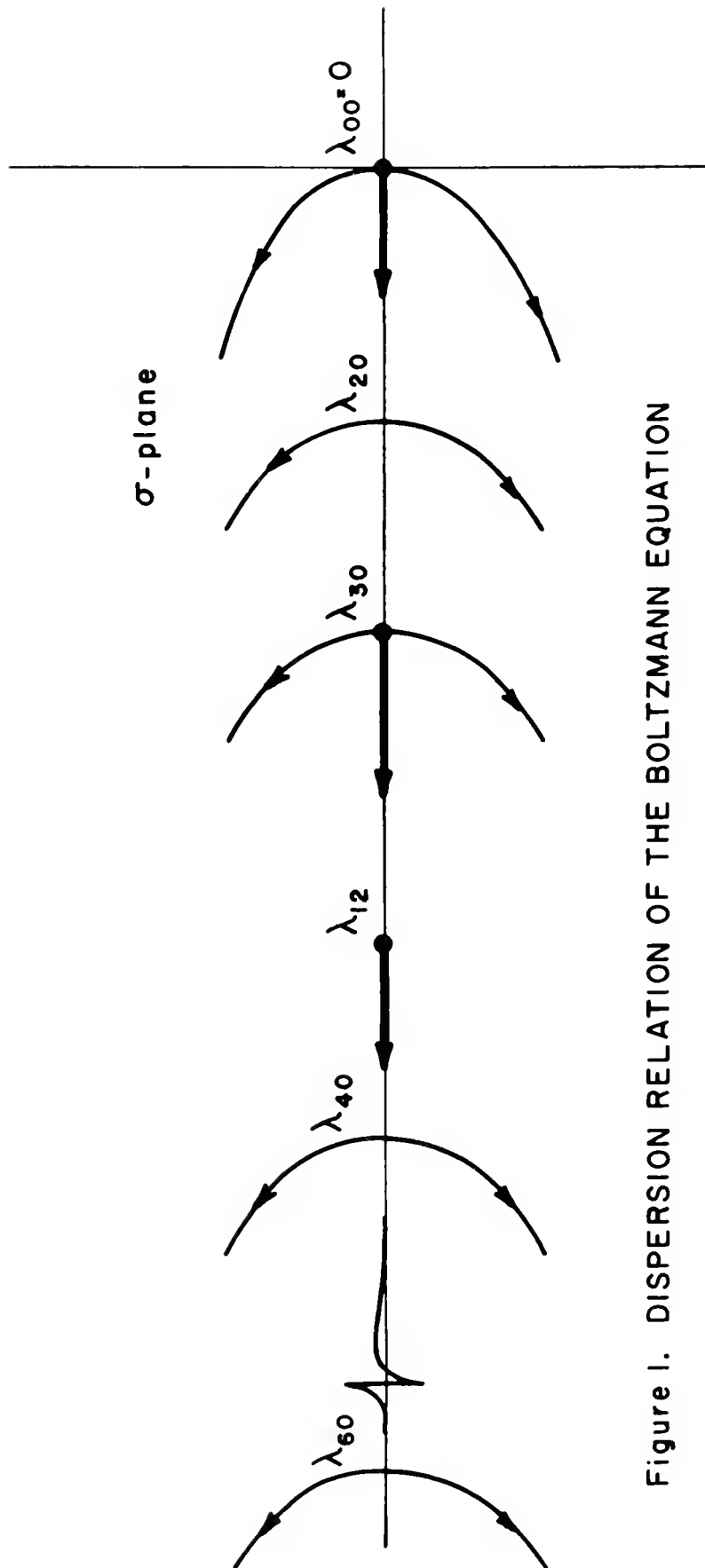


Figure 1. DISPERSION RELATION OF THE BOLTZMANN EQUATION

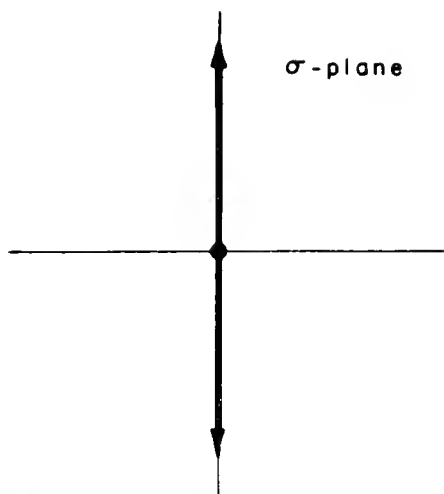


Figure 2. THE NON-DISSIPATIVE OR EULER DISPERSION LAW

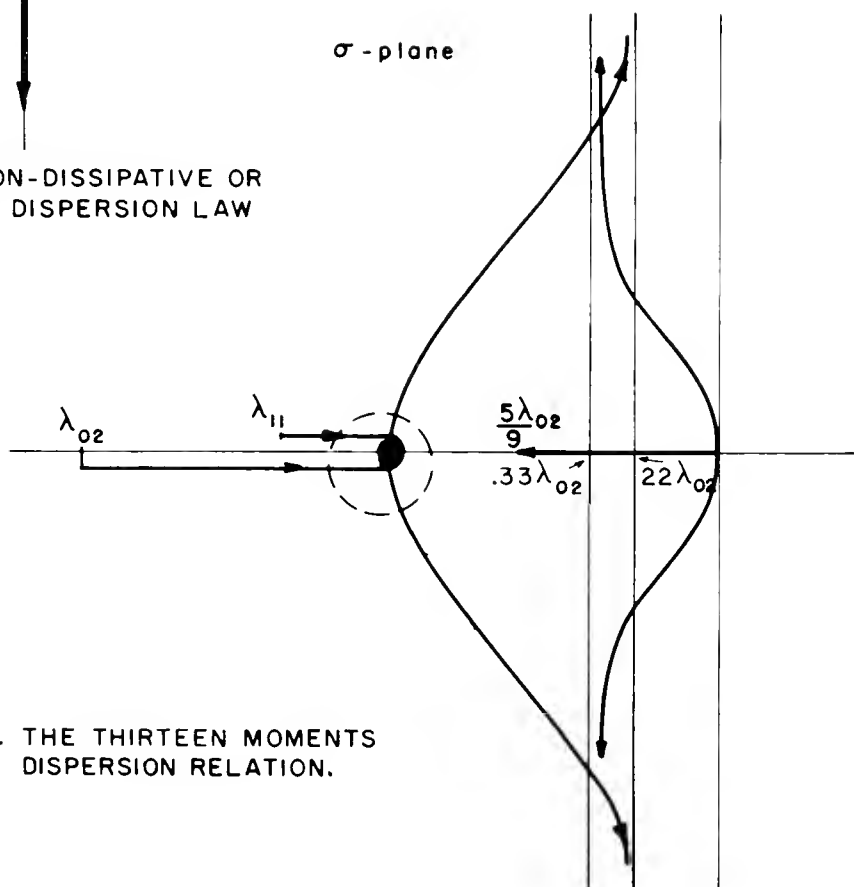


Figure 3. THE THIRTEEN MOMENTS DISPERSION RELATION.

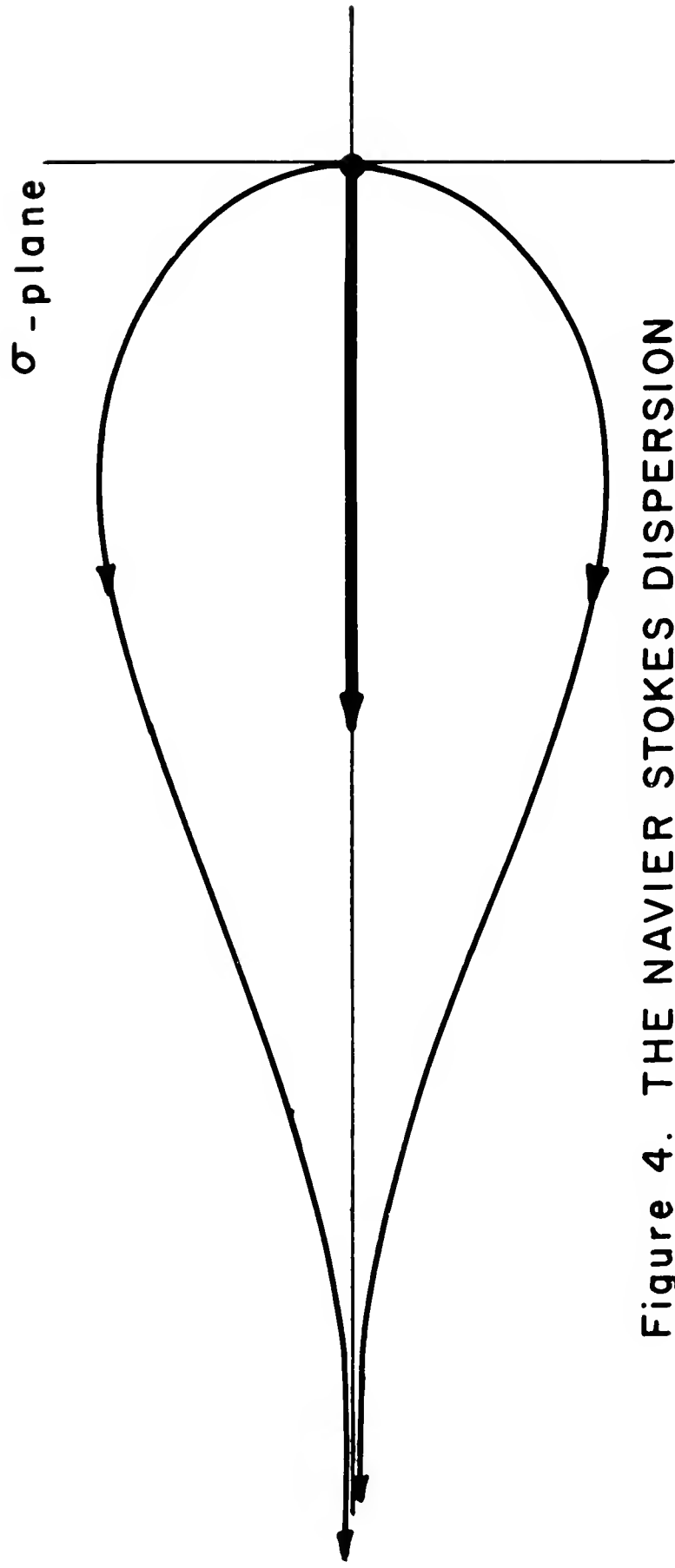


Figure 4. THE NAVIER STOKES DISPERSION
RELATION.

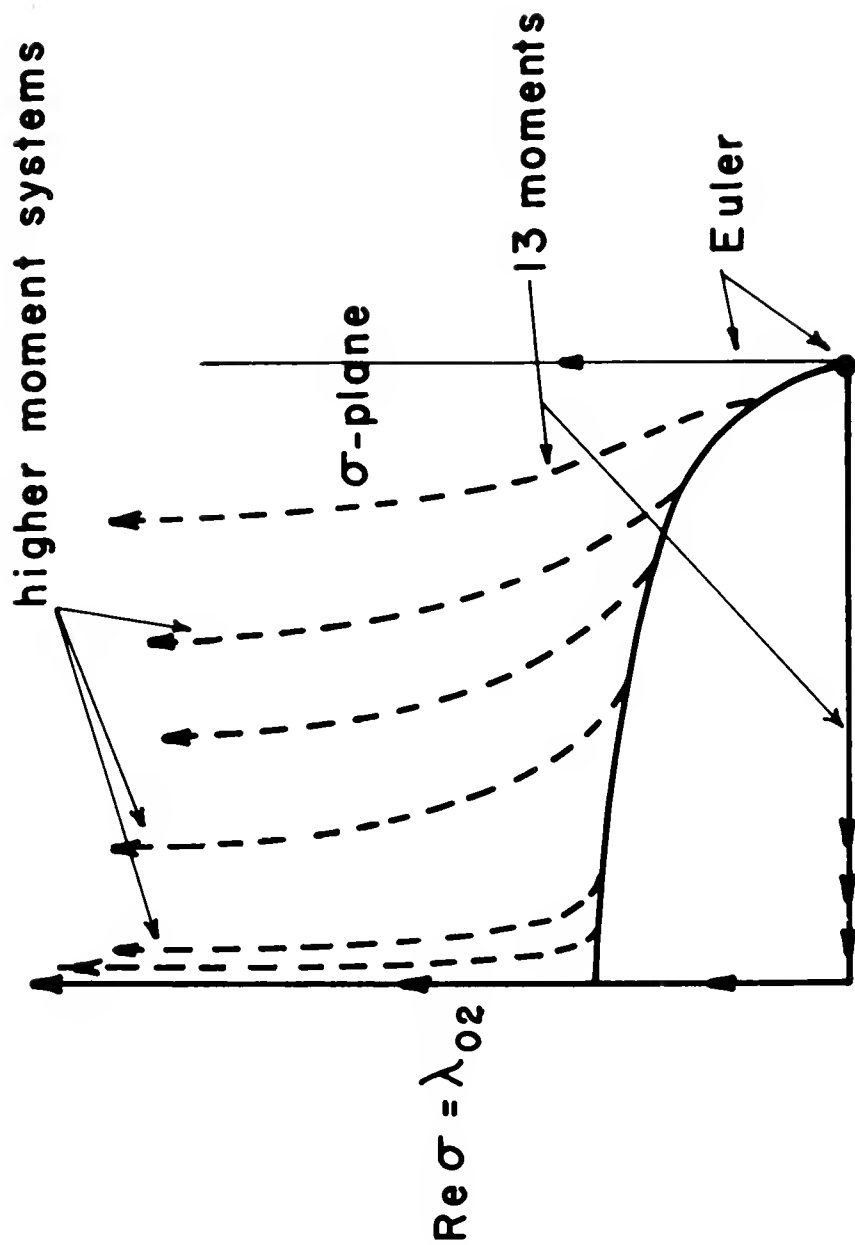


Figure 5. THE SINGLE RELAXATION DISPERSION LAW.

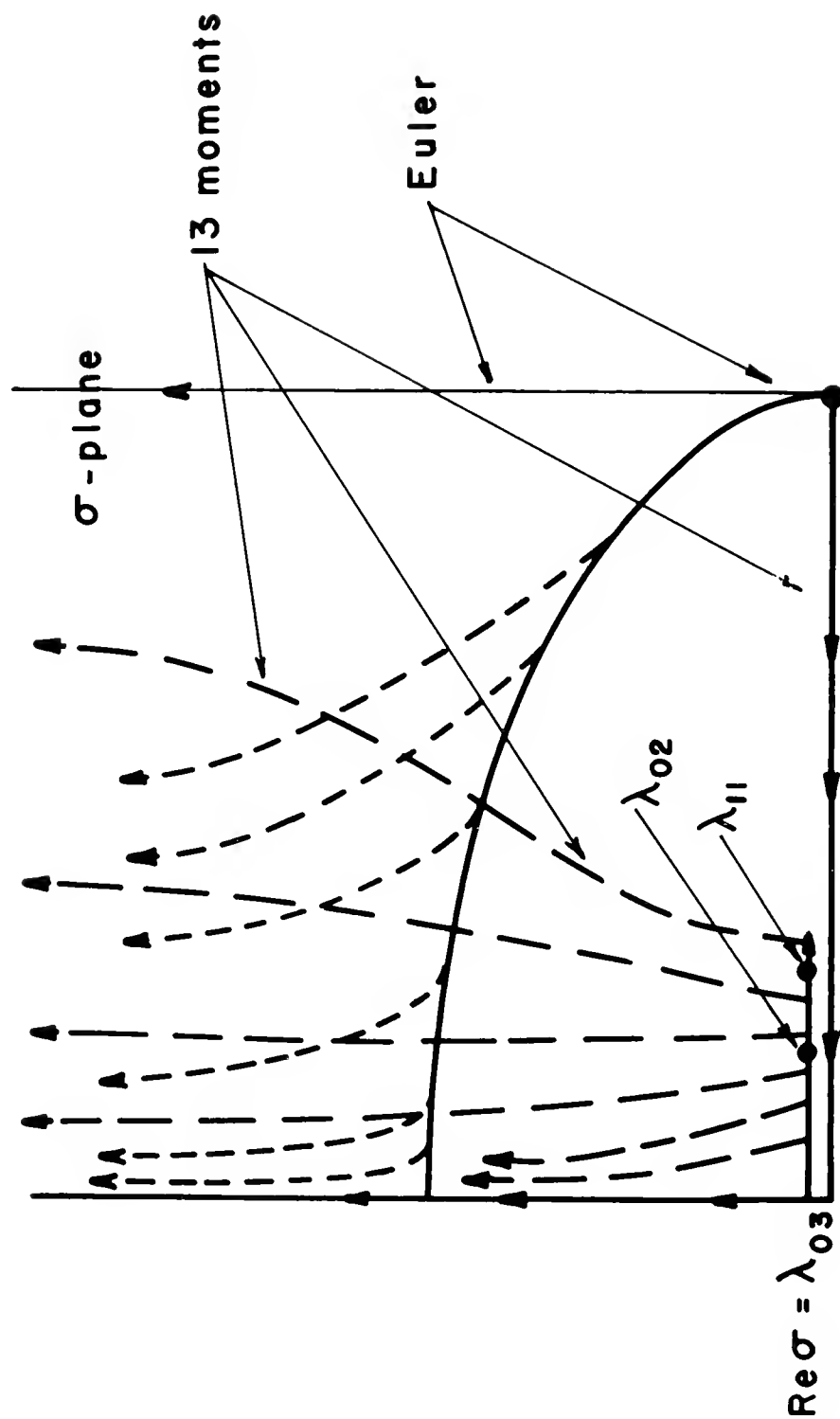


Figure 6. THE THREE RELAXATION DISPERSION LAW.

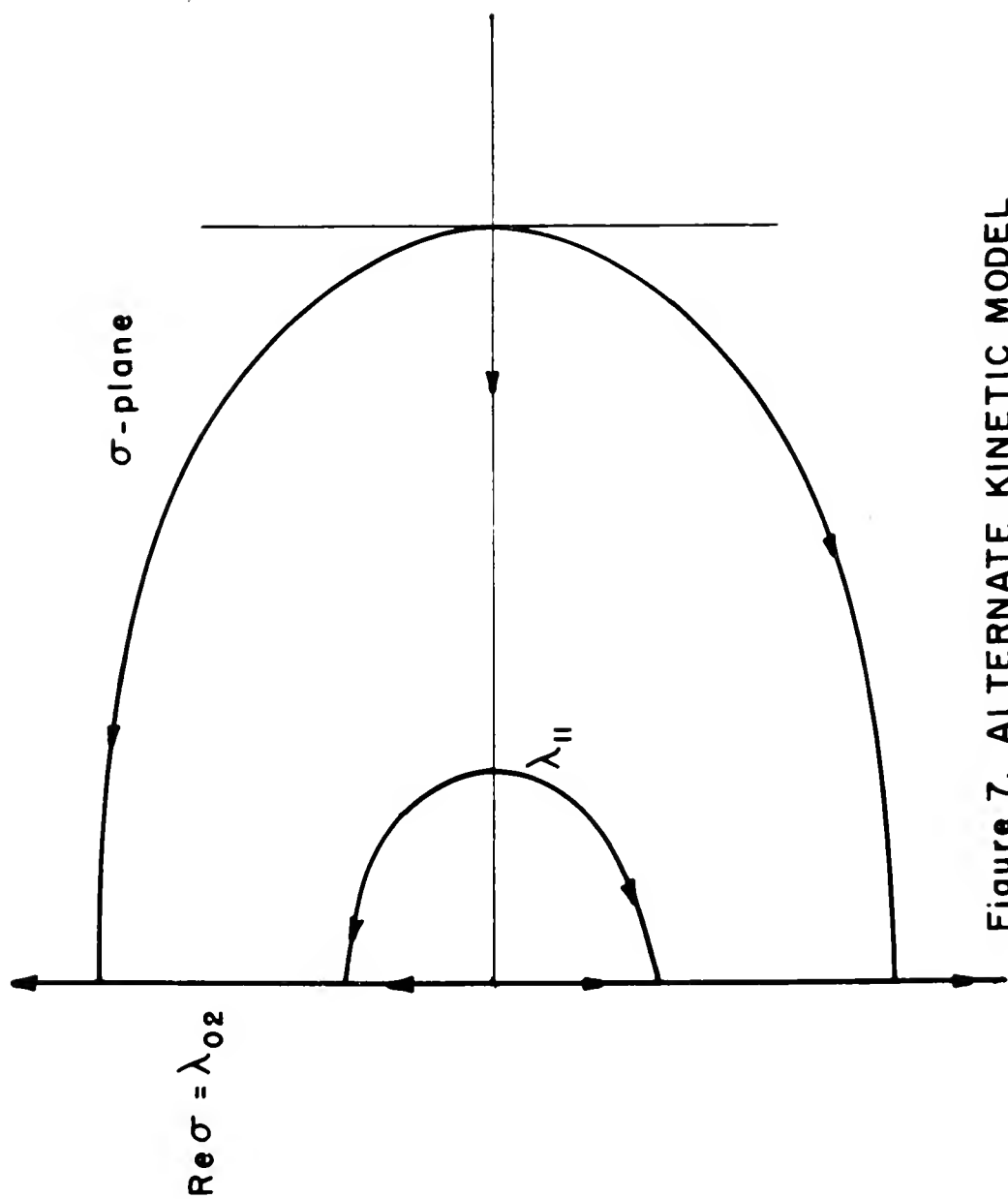


Figure 7. ALTERNATE KINETIC MODEL

4 Washington Place, New York 3, N. Y.

NOV 16 1967 DATE DUE

GAYLORO			PRINTED IN U.S.A.

NYU
MF-17

c.2

Sirovich

The initial value problem,
sound propagation,...

NYU
MF-17

c.2

Sirovich

The initial value problem,
sound propagation,....

**N. Y. U. Institute of
Mathematical Sciences**

4 Washington Place
New York 3, N. Y.

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY

4 Washington Place, New York 3, N. Y.